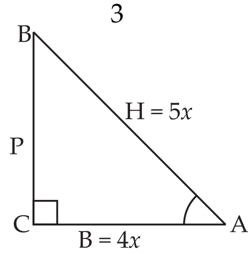
EXERCISE 8.1

Choose the correct answer from the given four options:

Q1. If $\cos A = \frac{4}{5}$, then the value of tan A is

(*a*) $\frac{3}{5}$ (*b*) $\frac{3}{4}$ (*c*) $\frac{4}{3}$ (*d*) $\frac{5}{3}$



Sol. (*b*): $\cos A = \frac{4}{5} = \frac{B}{H} = \frac{4x}{5x}$

$P^2 + B^2 = H^2$ (By Pythagoras theorem)

$\Rightarrow \quad P^2 + (4x)^2 = (5x)^2$

$\Rightarrow \quad P^2 = 25x^2 - 16x^2$

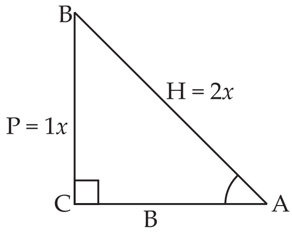
$\Rightarrow \quad P^2 = 9x^2$

$\Rightarrow \quad P = 3x$

$\therefore \quad \tan A = \frac{P}{B} = \frac{3x}{4x} = \frac{3}{4}$, which verifies option (*b*).

Q2. If $\sin A = \frac{1}{2}$, then the value of cot A is

(*a*) $\sqrt{3}$ (*b*) $\frac{1}{\sqrt{3}}$ (*c*) $\frac{\sqrt{3}}{2}$ (*d*) 1



Sol. (*a*): $\sin A = \frac{1}{2} = \frac{P}{H} = \frac{1x}{2x}$

$B^2 + P^2 = H^2$

$\Rightarrow \quad B^2 + (1x)^2 = (2x)^2$

$\Rightarrow \quad B^2 = 4x^2 - 1x^2$

$\Rightarrow \quad B^2 = 3x^2 \Rightarrow B = \sqrt{3}x$

$\therefore \quad \cot A = \frac{B}{P} = \frac{\sqrt{3}x}{1x} = \sqrt{3}$

Hence, right option is (*a*).

Q3. The value of the expression

$\operatorname{cosec}(75° + \theta) - \sec(15° - \theta) - \tan(55° + \theta) + \cot(35° - \theta)$ is

(*a*) –1 (*b*) 0 (*c*) 1 (*d*) $\frac{3}{2}$

Sol. (*b*): $(75° + \theta)$ and $(15° - \theta)$, are complements of each other. Similarly, $(55° + \theta)$ and $(35° - \theta)$ are also complements.

So, $\operatorname{cosec}(75° + \theta) - \sec(15° - \theta) - \tan(55° + \theta) + \cot(35° - \theta)$

$= \operatorname{cosec}[90° - (15° - \theta)] - \sec(15° - \theta) - \tan(55° + \theta) + \cot[90° - (55° + \theta)]$

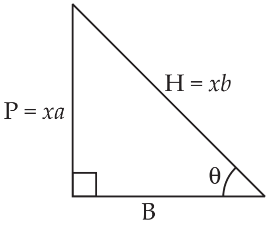
$= \sec(15° - \theta) - \sec(15° - \theta) - \tan(55° + \theta) + \tan(55° + \theta)$
$= 0$

Hence, right option is (*b*).

Q4. Given that $\sin\theta = \frac{a}{b}$, then $\cos\theta$ is equal to

(*a*) $\frac{b}{\sqrt{b^2 - a^2}}$ (*b*) $\frac{b}{a}$ (*c*) $\frac{\sqrt{b^2 - a^2}}{b}$ (*d*) $\frac{a}{\sqrt{b^2 - a^2}}$

Sol. (*c*): $\sin\theta = \frac{a}{b} = \frac{P}{H} = \frac{ax}{bx}$ [Given]



By Pythagoras theorem,

$$B^2 + P^2 = H^2$$
$$\Rightarrow B^2 + (ax)^2 = (bx)^2$$
$$\Rightarrow B^2 = b^2x^2 - a^2x^2$$
$$\Rightarrow B^2 = x^2(b^2 - a^2)$$
$$\Rightarrow B = x\sqrt{b^2 - a^2}$$
$$\therefore \cos\theta = \frac{B}{H} = \frac{x\sqrt{(b^2 - a^2)}}{bx} = \frac{\sqrt{b^2 - a^2}}{b},$$

which verifies the option (*c*).

Q5. If $\cos(\alpha + \beta) = 0$, then $\sin(\alpha - \beta)$ can be reduced to

(*a*) $\cos\beta$ (*b*) $\cos 2\beta$ (*c*) $\sin\alpha$ (*d*) $\sin 2\alpha$

Sol. (*b*): $\cos(\alpha + \beta) = 0$ [Given]

$$\Rightarrow \cos(\alpha + \beta) = \cos 90°$$
$$\Rightarrow \alpha + \beta = 90°$$
$$\Rightarrow \alpha = 90° - \beta$$

Now, $\sin(\alpha - \beta) = \sin(90° - \beta - \beta)$
$= \sin(90° - 2\beta)$

$$\Rightarrow \sin(\alpha - \beta) = \cos 2\beta$$

Hence, verifies the option (*b*).

Q6. The value of ($\tan 1° \tan 2° \tan 3° \ldots \tan 89°$) is

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) $\frac{1}{2}$

Sol. (*b*): ($\tan 1° \tan 2° \tan 3° \ldots \tan 89°$)

$= (\tan 1° \tan 89°)(\tan 2° \tan 88°)(\tan 3° \tan 87°) \ldots (\tan 45° \tan 45°)$
$= [\tan 1° \tan(90° - 1)][\tan 2° \tan(90° - 2)][\tan 3° \tan(90° - 3)] \ldots \tan 45° \tan(90° - 45°)$
$= \tan 1° \cot 1° \tan 2° \cot 2° \tan 3° \cot 3° \ldots \tan 45° \cot 45°$
$= \tan 1° \times \frac{1}{\tan 1°} \tan 2° . \frac{1}{\tan 2°} \tan 3° \cdot \frac{1}{\tan 3°} \cdots \frac{\tan 45°}{\tan 45°}$
$= 1 \cdot 1 \cdot 1 \cdot 1. \ldots 1 \cdot 1$
$= 1$

Hence, verifies the option (*b*).

Q7. If $\cos 9\alpha = \sin \alpha$ and $9\alpha < 90°$, then the value of $\tan 5\alpha$ is

(*a*) $\frac{1}{\sqrt{3}}$ (*b*) $\sqrt{3}$ (*c*) 1 (*d*) 0

Sol. (*c*): $\cos 9\alpha = \sin \alpha$

$\Rightarrow \cos 9\alpha = \cos (90° - \alpha)$

$\Rightarrow 9\alpha = 90° - \alpha$

$\Rightarrow 10\alpha = 90°$

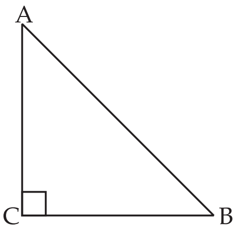
$\Rightarrow \alpha = 9°$

$\therefore \tan 5\alpha = \tan 5 \times 9° = \tan 45° = 1$

Hence, verifies the option (*c*).

Q8. If ΔABC is right angled at C, then the value of $\cos (A + B)$ is

(*a*) 0 (*b*) 1 (*c*) $\frac{1}{2}$ (*d*) $\frac{\sqrt{3}}{2}$



Sol. (*a*): $\angle A + \angle B + \angle C = 180°$

(Angle sum property of a triangle)

$\Rightarrow (A + B) + 90° = 180°$

$\Rightarrow \angle A + \angle B = 90°$

$\Rightarrow \cos (A + B) = \cos 90° = 0$

Hence, verifies the option (*a*).

Q9. If $\sin A + \sin^2 A = 1$, then the value of the expression $(\cos^2 A + \cos^4 A)$ is

(*a*) 1 (*b*) $\frac{1}{2}$ (*c*) 2 (*d*) 3

Sol. (*a*): $\sin A + \sin^2 A = 1$ [Given]

$\Rightarrow \sin A = (1 - \sin^2 A)$

$\Rightarrow \sin A = \cos^2 A$

$\Rightarrow \sin^2 A = \cos^4 A$ [Squaring both sides]

$\Rightarrow 1 - \cos^2 A = \cos^4 A$ [$\because \sin^2 A = 1 - \cos^2 A$]

$\Rightarrow 1 = \cos^2 A + \cos^4 A$

Hence, verifies the option (*a*).

Q10. Given that $\sin \alpha = \frac{1}{2}, \cos \beta = \frac{1}{2},$ then value of $(\alpha + \beta)$ is

(*a*) 0° (*b*) 30° (*c*) 60° (*d*) 90°

Sol. (*d*): $\sin \alpha = \frac{1}{2}$ [Given]

$\Rightarrow \sin \alpha = \sin 30°$

$\Rightarrow \alpha = 30°$

Also, $\cos \beta = \frac{1}{2}$

$\Rightarrow \cos \beta = \cos 60°$

$\Rightarrow \beta = 60°$

$\therefore \alpha + \beta = 30° + 60° = 90°$

Hence, verifies the option (*d*).

Q11. The value of the expression

$$\left[\frac{\sin^2 22° + \sin^2 68°}{\cos^2 22° + \cos^2 68°} + \sin^2 63° + \cos 63° \sin 27°\right] \text{ is}$$

(*a*) 3 (*b*) 2 (*c*) 1 (*d*) 0

Sol. (*b*): Here, the complement angle of each angle is available. So, by using formula for complementary angles, we get

$$\frac{\sin^2 22° + \sin^2(90° - 22°)}{\cos^2 22° + \cos^2(90° - 22°)} + \sin^2 63° + \cos 63° \sin(90° - 63°)$$

$$\Rightarrow \frac{\sin^2 22° + \cos^2 22°}{\cos^2 22° + \sin^2 22°} + \sin^2 63° + \cos 63° \cos 63°$$

$$\Rightarrow \frac{1}{1} + \sin^2 63° + \cos^2 63° \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$\Rightarrow 1 + 1 = 2$$

Hence, verifies the option (*b*).

Q12. If $4 \tan\theta = 3$, then $\left[\frac{4\sin\theta - \cos\theta}{4\sin\theta + \cos\theta}\right]$ is equal to

(*a*) $\frac{2}{3}$ (*b*) $\frac{1}{3}$ (*c*) $\frac{1}{2}$ (*d*) $\frac{3}{4}$

Sol. (*c*): $\frac{4\sin\theta - \cos\theta}{4\sin\theta + \cos\theta}$

$$= \frac{4\left(\frac{\sin\theta}{\cos\theta}\right) - 1}{4\left(\frac{\sin\theta}{\cos\theta}\right) + 1}$$ (Dividing the numerator and denominator throughout by $\cos\theta$)

$$= \frac{4\tan\theta - 1}{4\tan\theta + 1} = \frac{3-1}{3+1} \qquad [\because 4\tan\theta = 3]$$

$= \frac{2}{4} = \frac{1}{2}$. Hence, verifies the option (*c*).

Q13. If $\sin\theta - \cos\theta = 0$, then the value of $(\sin^4\theta + \cos^4\theta)$ is

(*a*) 1 (*b*) $\frac{3}{4}$ (*c*) $\frac{1}{2}$ (*d*) $\frac{1}{4}$

Sol. (*c*): $\sin\theta - \cos\theta = 0$ [Given]

$\Rightarrow \sin\theta = \cos\theta$

$\Rightarrow \sin\theta = \sin(90° - \theta)$

$\Rightarrow \theta = 90° - \theta$

$\Rightarrow 2\theta = 90°$

$\Rightarrow \theta = 45°$

Now, $\sin^4\theta + \cos^4\theta = (\sin 45°)^4 + (\cos 45°)^4$

$$= \left(\frac{1}{\sqrt{2}}\right)^4 + \left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

Hence, verifies the option (*c*).

Q14. sin (45° + θ) – cos (45° – θ) is equal to

(*a*) 2 cos θ (*b*) 0 (*c*) 2 sin θ (*d*) 1

Sol. (*b*): (45° + θ) and (45° – θ) are complementary angles.

∴ By using formulae of complementary angles,

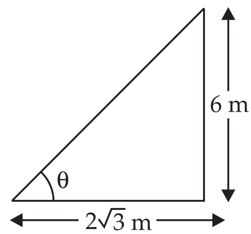
sin (45° + θ) – cos (45° – θ)

$$= \sin(45° + \theta) - \cos[90° - (45° + \theta)]$$
$$= \sin(45° + \theta) - \sin(45° + \theta)$$
$$= 0$$

Hence, verifies the option (*b*).

Q15. If a pole 6 m high casts a shadow of $2\sqrt{3}$ m long on the ground, then the Sun's elevation is

(*a*) 60° (*b*) 45°
(*c*) 30° (*d*) 90°



Sol. (*a*): $\tan\theta = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \frac{\sqrt{3}}{1}$

$\Rightarrow$ $\tan\theta = \tan 60°$

$\Rightarrow$ $\theta = 60°$

Hence, the right option is (*a*).

EXERCISE 8.2

Write True or False and justify your answer in each of the following:

Q1. $\dfrac{\tan 47^\circ}{\cot 43^\circ} = 1$

Sol. True: 47° and 43° are complementary angles.

$$\therefore \quad \frac{\tan 47^\circ}{\cot 43^\circ} = \frac{\tan 47^\circ}{\cot (90^\circ - 47^\circ)} = \frac{\tan 47^\circ}{\tan 47^\circ} = 1$$

Hence, the given expression is true.

Q2. The value of the expression $(\cos^2 23^\circ - \sin^2 67^\circ)$ is positive.

Sol. False: 23° and 67° are complementary angles so

$$\begin{aligned} \cos^2 23^\circ - \sin^2 67^\circ &= \cos^2 23^\circ - \sin^2 (90^\circ - 23^\circ) \\ &= \cos^2 23^\circ - \cos^2 23^\circ \\ &= 0 \end{aligned}$$

So, the value of the given expression is not positive. Hence, the given statement is false.

Q3. The value of the expression $(\sin 80^\circ - \cos 80^\circ)$ is negative.

Sol. False: 80° is near to 90°, $\sin 90^\circ = 1$ and $\cos 90^\circ = 0$

So, the given expression $\sin 80^\circ - \cos 80^\circ > 0$

So, the value of the given expression is positive. So, the given statement is false.

Q4. $\sqrt{(1-\cos^2\theta)\sec^2\theta} = \tan\theta$

Sol. True: $\quad \text{LHS} = \sqrt{(1-\cos^2\theta)\sec^2\theta} = \sqrt{\sin^2\theta \cdot \dfrac{1}{\cos^2\theta}}$

$$= \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} = \tan\theta = \text{RHS}$$

Hence, the given expression is true.

Q5. If $\cos A + \cos^2 A = 1$, then $\sin^2 A + \sin^4 A = 1$

Sol. True:

$\cos A + \cos^2 A = 1$ [Given]

$\Rightarrow \quad \cos A = 1 - \cos^2 A$

$\Rightarrow \quad \cos A = \sin^2 A$

$\Rightarrow \quad \cos^2 A = \sin^4 A$

Now, $\quad \text{LHS} = \sin^2 A + \sin^4 A$

$= \cos A + \cos^2 A$

$= 1 = \text{RHS}$

Hence, the given statement is true.

Q6. $(\tan\theta + 2)(2\tan\theta + 1) = 5\tan\theta + \sec^2\theta$

Sol. False:

$$\begin{aligned} \text{LHS} &= (\tan\theta + 2)(2\tan\theta + 1) \\ &= \tan\theta(2\tan\theta + 1) + 2(2\tan\theta + 1) \\ &= 2\tan^2\theta + \tan\theta + 4\tan\theta + 2 \\ &= 2\tan^2\theta + 5\tan\theta + 2 \\ &= 2(\tan^2\theta + 1) + 5\tan\theta \\ &= 2\sec^2\theta + 5\tan\theta \neq \text{RHS} \end{aligned}$$

Hence, the given statement is false.

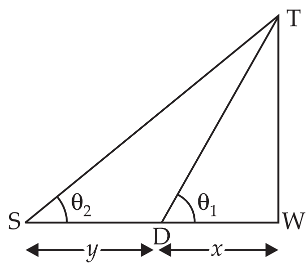
Q7. If the length of the shadow of a tower is increasing, then the angle of elevation of the sun is also increasing.

Sol. False: The shadow of a tower on the ground increases from x to $(x + y)$ when angle of elevation of the sun changes from θ_1 to θ_2.

$\therefore$ θ_1 is the exterior angle of ΔTSD

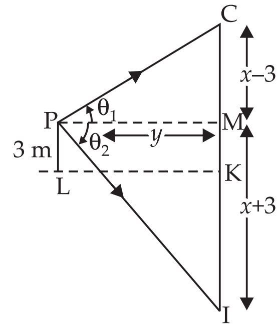
so $\quad \theta_1 > \theta_2$

So, on increasing the length of shadow the angle of elevation decreases.



Hence, the given statement is false.

Q8. If a man standing on a platform 3 m above the surface of a lake observes a cloud and its reflection in the lake, then the angle of elevation of the cloud is equal to the angle of depression of its reflection.



Sol. False: The observer is at the platform (P) 3 m above the surface LK of the lake.
He observes the angle of elevation of cloud C from P and its reflection image in the lake is formed at I. The observer measures the angle of depression of image (I) θ_2. Draw PM $\perp$ on the vertical line passing through the cloud and its image.

CK = KI = x by the prop. of reflection

$$CM = CK - MK = x - 3$$
$$MI = KI + MK = x + 3$$

Now, $\tan\theta_1 = \dfrac{x-3}{y}$ and $\tan\theta_2 = \dfrac{x+3}{y}$

$\Rightarrow$ $y = \dfrac{x-3}{\tan\theta_1}$ and $y = \dfrac{x+3}{\tan\theta_2}$

$\Rightarrow$ $$\frac{x+3}{\tan\theta_2} = \frac{x-3}{\tan\theta_1}$$

$\Rightarrow$ $$\tan\theta_2 = \left(\frac{x+3}{x-3}\right)\tan\theta_1$$

$\Rightarrow$ $$\tan\theta_1 \neq \tan\theta_2$$

or $$\theta_1 \neq \theta_2$$

Alternate Method: By the property of image formation, the distance of image and the object are equal from the reflecting surface.

So, $$KC = KI$$

$\Rightarrow$ $$MI \neq MC$$

$\Rightarrow$ $$\Delta MPC \not\cong \Delta MPI$$

so $$\theta_1 \neq \theta_2$$

Q9. The value of 2 sin θ can be $\left(a + \dfrac{1}{a}\right)$, where a is a positive number, and $a \neq 1$.

Sol. False: Consider 'a' and '$\dfrac{1}{a}$' as positive numbers and $a \neq 0$

Arithmetic mean (AM) of a and $\dfrac{1}{a}$ = $\dfrac{\left(a + \dfrac{1}{a}\right)}{2}$

Geometric mean (GM) of a and $\dfrac{1}{a}$ = $\sqrt{a \times \dfrac{1}{a}} = 1$

$\because$ $$AM > GM$$

$\therefore$ $$\frac{\left(a + \dfrac{1}{a}\right)}{2} > 1$$

$\Rightarrow \qquad \left(a+\dfrac{1}{a}\right) > 2$

Let $\qquad a+\dfrac{1}{a} = 2\sin\theta$

$\Rightarrow \qquad 2\sin\theta > 2$

$\Rightarrow \qquad \sin\theta > 1$

which can never be possible.

Hence, our consideration that $a+\dfrac{1}{a} = 2\sin\theta$ is false.

Alternate Method: a is positive and $a \neq 1$ *i.e.*, a can be above 0 to all real and values except 1.

Let $0 < a < 1$ then $\dfrac{1}{a}$ will be more than 1 so $a+\dfrac{1}{a} > 2$ for any value of a.

Let $\qquad a = 0.2 \Rightarrow a+\dfrac{1}{a} = 0.2+\dfrac{1}{0.2} = 5.2$

$\qquad a = 0.9 \Rightarrow a+\dfrac{1}{a} = 0.9+\dfrac{1}{0.9} = 0.9+1.111 = 2.011$

Put $a+\dfrac{1}{a} = 2\sin\theta$

$\therefore \qquad 2\sin\theta > 2$

$\Rightarrow \qquad \sin\theta > 1$

which is impossible so $2\sin\theta \neq a+\dfrac{1}{a}$

Hence, the given statement is false. If we take any value of a more than one, then the value of $a+\dfrac{1}{a}$ is always greater than 2 which repeats the result.

Q10. $\cos\theta = \dfrac{a^2+b^2}{2ab}$, where a and b are two distinct numbers such that $ab > 0$.

Sol. False: Consider two numbers a^2 and b^2 then

Arithmetic mean (AM) of a^2 and b^2 = $\dfrac{a^2+b^2}{2}$

Geometric mean (GM) of a^2 and b^2 = $\sqrt{a^2 \times b^2}$

$\Rightarrow \qquad \text{GM} = ab$

$\because \qquad \text{AM} > \text{GM}$

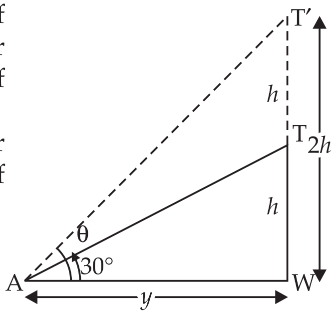
$\therefore \qquad \dfrac{a^2+b^2}{2} > ab$

$\Rightarrow \qquad \dfrac{a^2+b^2}{2ab} > 1$

$$\Rightarrow \qquad \cos\theta > 1 \qquad \left(\text{Given } \cos\theta = \frac{a^2+b^2}{2ab}\right)$$

But, the value of cos θ can never be greater than 1.
So, the given expression is false.

Q11. The angle of elevation of the top of a tower is 30°. If the height of the tower is doubled, then the angle of elevation of its top will also be doubled.



Sol. False: Let the height of the tower is h. For the observer at A the angle of elevation is equal to 30°.

$$\tan 30° = \frac{h}{y}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{h}{y}$$

$$\Rightarrow \quad y = h\sqrt{3}$$

Now, the height of the tower increases to $2h$.
Now, let the new angle of elevation at A becomes θ then

$$\tan\theta = \frac{2h}{y}$$

$$\Rightarrow \quad \tan\theta = \frac{2h}{h\sqrt{3}}$$

$$\Rightarrow \quad \tan\theta = \frac{2}{\sqrt{3}}$$

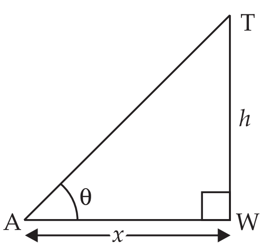
But, $\tan 60° = \sqrt{3}$

$$\Rightarrow \quad \tan 60° = \sqrt{3} \neq \frac{2}{\sqrt{3}}$$

So, $\theta \neq 60°$

Hence, angle of elevation will not be doubled or the given statement is false.

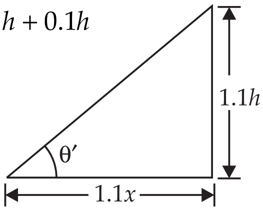
Q12. If the height of a tower and the distance of the point of observation from its foot, both are increased by 10%, then the angle of elevation of its top remains: unchanged.



Sol. True: Let height h of tower TW makes an angle of elevation θ to observer at A and the distance from foot of tower to the observer is x.

$$\therefore \quad \tan\theta = \frac{h}{x} \qquad \text{(I)}$$

Now, h and x increases by 10%



$\therefore \quad h' = h + 10\% \text{ of } h = h + \dfrac{10}{100} \times h = h + 0.1h$

$\Rightarrow \quad h' = 1.1\,h$

Similarly, $\quad x' = 1.1x$

$\therefore \quad \tan\theta' = \dfrac{1.1h}{1.1x} = \dfrac{h}{x} \quad$ (II)

From (I) and (II), we get

$\tan\theta = \tan\theta'$

$\Rightarrow \quad \theta = \theta'$

Hence, the given statement is true.

EXERCISE 8.3

Prove the following questions (from Q1 to Q7):

Q1. $\dfrac{\sin\theta}{(1+\cos\theta)} + \dfrac{(1+\cos\theta)}{\sin\theta} = 2\operatorname{cosec}\theta$

Sol. (*i*) If there is (+)ve or (–)ve sign in D^r and N^r of the expression, then keep the expression in brackets.

(*ii*) LHS is more difficult than RHS so we will start from LHS.

(*iii*) Use identities, if applicable.

(*iv*) Convert the expression into $\sin\theta$ and $\cos\theta$.

$$\text{LHS} = \frac{\sin\theta}{(1+\cos\theta)} + \frac{(1+\cos\theta)}{\sin\theta}$$

$$= \frac{\sin^2\theta + (1+\cos\theta)^2}{(1+\cos\theta)\sin\theta}$$

$$= \frac{\sin^2\theta + (1)^2 + (\cos\theta)^2 + 2(1)(\cos\theta)}{\sin\theta\,(1+\cos\theta)}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{1+1+2\cos\theta}{\sin\theta\,(1+\cos\theta)} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{2+2\cos\theta}{\sin\theta\,(1+\cos\theta)} = \frac{2\,(1+\cos\theta)}{\sin\theta\,(1+\cos\theta)}$$

$$= \frac{2}{\sin\theta}$$

$$= 2\operatorname{cosec}\theta$$

$\therefore$ LHS = RHS

Hence, proved.

Q2. $\dfrac{\tan A}{1+\sec A} - \dfrac{\tan A}{1-\sec A} = 2 \operatorname{cosec} A$

Sol. We will start from LHS.

[We can take tan A as common and (1 – sec A) (1 + sec A) makes $(1 - \sec^2 A = -\tan^2 A)$ so formulae will be used.]

$$\text{LHS} = \frac{\tan A}{(1+\sec A)} - \frac{\tan A}{(1-\sec A)}$$

$$= \tan A\left[\frac{1}{(1+\sec A)} - \frac{1}{(1-\sec A)}\right] \qquad \text{[Taking tan A common]}$$

$$= \tan A\left[\frac{1-\sec A-(1+\sec A)}{(1+\sec A)(1-\sec A)}\right] \qquad \text{[Taking LCM]}$$

$$= \tan A\left[\frac{-2\sec A}{(1-\sec^2 A)}\right] = \frac{\tan A(-2\sec A)}{-(\sec^2 A - 1)}$$

$$= \frac{-\tan A\, 2\sec A}{-\tan^2 A} = \frac{2\sec A}{\tan A}$$

$$= \frac{2\times\dfrac{1}{\cos A}}{\dfrac{\sin A}{\cos A}}$$

$$= \frac{2}{\sin A} = 2\operatorname{cosec} A = \text{RHS}$$

Hence, proved.

Q3. If $\tan A = \dfrac{3}{4}$, then $\sin A \cos A = \dfrac{12}{25}$.

Sol. $\tan A = \dfrac{3}{4}$ [Given]

$\Rightarrow \tan A = \dfrac{P}{B} = \dfrac{3x}{4x}$

$H^2 = P^2 + B^2$ [By Pythagoras theorem]

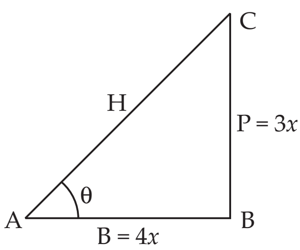
$= (3x)^2 + (4x)^2 = 9x^2 + 16x^2$

$\Rightarrow H^2 = 25x^2$

$\Rightarrow H = 5x$

$\therefore \sin A = \dfrac{P}{H} = \dfrac{3x}{5x} = \dfrac{3}{5}$

and $\cos A = \dfrac{B}{H} = \dfrac{4x}{5x} = \dfrac{4}{5}$



$$\therefore \quad \text{LHS} = \sin A \cos A$$
$$= \frac{3}{5} \times \frac{4}{5} = \frac{12}{25}$$
$$= \text{RHS}$$

Hence, verified.

Q4. $(\sin \alpha + \cos \alpha)(\tan \alpha + \cot \alpha) = \sec \alpha + \operatorname{cosec} \alpha$

Sol. $\text{LHS} = (\sin \alpha + \cos \alpha)(\tan \alpha + \cot \alpha)$

$$= (\sin \alpha + \cos \alpha)\left(\frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\sin \alpha}\right)$$
$$= (\sin \alpha + \cos \alpha)\left[\frac{\sin^2 \alpha + \cos^2 \alpha}{\sin \alpha \cos \alpha}\right]$$
$$= (\sin \alpha + \cos \alpha)\frac{1}{\sin \alpha \cos \alpha} \qquad [\because \sin^2 \alpha + \cos^2 \alpha = 1]$$
$$= \frac{\sin \alpha}{\sin \alpha \cos \alpha} + \frac{\cos \alpha}{\sin \alpha \cos \alpha}$$
$$= \frac{1}{\cos \alpha} + \frac{1}{\sin \alpha}$$
$$= \sec \alpha + \operatorname{cosec} \alpha$$
$$= \text{RHS}$$

Hence, proved.

Q5. $(\sqrt{3} + 1)(3 - \cot 30°) = \tan^3 60° - 2 \tan 60°$

Sol. $\text{LHS} = (\sqrt{3} + 1)(3 - \cot 30°)$

$$= (\sqrt{3} + 1)(3 - \sqrt{3})$$
$$= \sqrt{3}(3 - \sqrt{3}) + 1(3 - \sqrt{3})$$
$$= 3\sqrt{3} - 3 + 3 - \sqrt{3}$$
$$= 2\sqrt{3}$$

$$\text{RHS} = \tan^3 60° - 2 \sin 60°$$
$$= (\sqrt{3})^3 - 2 \times \frac{\sqrt{3}}{2}$$
$$= 3\sqrt{3} - \sqrt{3}$$
$$= 2\sqrt{3}$$

$$\Rightarrow \quad \text{LHS} = \text{RHS}$$

Hence, proved.

Q6. $1 + \frac{\cot^2 \alpha}{(1 + \operatorname{cosec} \alpha)} = \operatorname{cosec} \alpha$

Sol. $\text{LHS} = 1 + \frac{\cot^2 \alpha}{(1 + \operatorname{cosec} \alpha)}$

$$\therefore \qquad = 1 + \frac{(\text{cosec}^2 \alpha - 1)}{(1 + \text{cosec}\, \alpha)} \qquad [\because \cot^2 \alpha = \text{cosec}^2 \alpha - 1]$$

$$= 1 + \frac{(\text{cosec}\, \alpha - 1)(\text{cosec}\, \alpha + 1)}{(\text{cosec}\, \alpha + 1)} \qquad [\because a^2 - b^2 = (a-b)(a+b)]$$

$$= 1 + \text{cosec}\, \alpha - 1$$

$$= \text{cosec}\, \alpha$$

$$= \text{RHS}$$

Hence, proved.

Q7. $\tan\theta + \tan(90° - \theta) = \sec\theta \sec(90° - \theta)$

Sol. $\text{LHS} = \tan\theta + \tan(90° - \theta)$

$$= \tan\theta + \cot\theta$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \text{cosec}\,\theta \,.\, \sec\theta$$

$\Rightarrow \qquad \text{LHS} = \text{cosec}\,\theta \cdot \sec\theta$

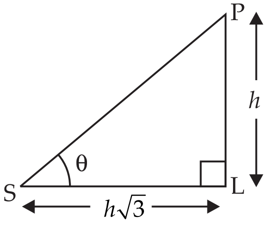
$\text{RHS} = \sec\theta \sec(90° - \theta)$

$$= \sec\theta\, \text{cosec}\,\theta$$

$\Rightarrow \qquad \text{LHS} = \text{RHS}$

Hence, verified.

Q8. Find the angle of elevation of the sun when the shadow of a pole h m high is $\sqrt{3}h$ m long.



Sol. Height of pole $= \text{PL} = h$

Length of shadow $= \text{SL} = h\sqrt{3}$

$\therefore \qquad \tan\theta = \dfrac{h}{h\sqrt{3}}$

$$= \frac{1}{\sqrt{3}}$$

$\Rightarrow \qquad \tan\theta = \tan 30°$

$\Rightarrow \qquad \theta = 30°$

Hence, the angle of elevation of sun is 30°.

Q9. If $\sqrt{3}\tan\theta = 1$, then find the value of $\sin^2\theta - \cos^2\theta$.

Sol. $\sqrt{3}\tan\theta = 1$ [Given]

$\Rightarrow \qquad \tan\theta = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \qquad \tan\theta = \tan 30°$

$\Rightarrow \quad \theta = 30°$

So, $\sin^2\theta - \cos^2\theta = \sin^2 30° - \cos^2 30°$ [$\because \quad \theta = 30°$]

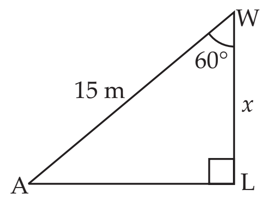
$$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - \frac{3}{4} = \frac{-2}{4} = \frac{-1}{2}$$

Q10. A ladder 15 m long just reaches the top of a vertical wall. If the ladder makes an angle of 60° with the wall, find the height of the wall.

Sol. Consider the vertical wall WL = x m (let)

Length of inclined ladder AW = 15 m [Given]

Ladder makes an angle of 60° with the wall.



$\therefore \quad \frac{x}{15} = \cos 60°$

$\Rightarrow \quad \frac{x}{15} = \frac{1}{2}$

$\Rightarrow \quad x = \frac{15}{2} = 7.5$

Hence, the height of the wall = 7.5 m.

Q11. Simplify: $(1 + \tan^2\theta)(1 - \sin\theta)(1 + \sin\theta)$

Sol. $(1 + \tan^2\theta)(1 - \sin\theta)(1 + \sin\theta)$

$$= (\sec^2\theta)(1 - \sin^2\theta)$$
$$= \sec^2\theta \cos^2\theta$$
$$= \frac{1}{\cos^2\theta} \times \cos^2\theta$$
$$= 1$$

Q12. If $2\sin^2\theta - \cos^2\theta = 2$, then find the value of θ.

Sol. Given: $2\sin^2\theta - \cos^2\theta = 2$

$\Rightarrow \quad 2\sin^2\theta - (1 - \sin^2\theta) = 2$

$\Rightarrow \quad 2\sin^2\theta - 1 + \sin^2\theta = 2$

$\Rightarrow \quad 3\sin^2\theta = 2 + 1$

$\Rightarrow \quad \sin^2\theta = \frac{3}{3} = 1$

$\Rightarrow \quad \sin\theta = +1, -1$

$\Rightarrow \quad \sin\theta = 1$

$\Rightarrow \quad \sin\theta = \sin 90°$

$\Rightarrow \quad \theta = 90°$

$\Rightarrow \quad \sin\theta = -1$

$\Rightarrow \quad \sin\theta = -\sin 90°$

$\Rightarrow \quad -\sin(-\theta) = -\sin 90°$

$\Rightarrow \quad \sin(-\theta) = \sin 90°$

$\Rightarrow \quad -\theta = 90°$

$\Rightarrow \quad \theta = -90°$

i.e., $\quad \theta = 360° - 90° = 270°$

Hence, $\theta = 90°$ and $270°$

Q13. Show that $\dfrac{\cos^2(45° + \theta) + \cos^2(45° - \theta)}{\tan(60° + \theta)\tan(30° - \theta)} = 1.$

Sol. $(45° + \theta)$, $(45° - \theta)$ and $(60° + \theta)$, $(30° - \theta)$ are complementary angles so by using complementary angle formulae, we get

$$\frac{\cos^2(45° + \theta) + \cos^2(45° - \theta)}{\tan(60° + \theta)\tan(30° - \theta)}$$

$$= \frac{\cos^2(45° + \theta) + \cos^2[90° - (45° + \theta)]}{\tan(60° + \theta)\tan[90° - (60° + \theta)]}$$

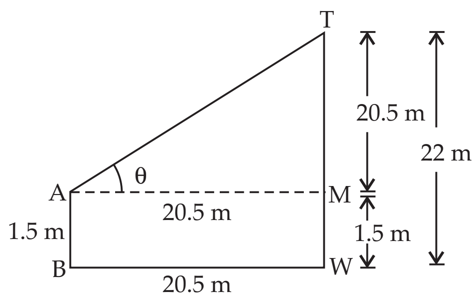
$$[\because \quad 90° - (45° + \theta) = 90° - 45° - \theta = 45° - \theta]$$

$$= \frac{\cos^2(45° + \theta) + \sin^2(45° + \theta)}{\tan(60° + \theta) \cdot \cot(60° + \theta)}$$

$$= \frac{1}{\tan(60° + \theta) \cdot \dfrac{1}{\tan(60° + \theta)}} = \frac{1}{1} = 1 = \text{RHS}$$

Hence, proved.

Q14. An observer 1.5 m tall is 20.5 m away from a tower 22 m high. Determine the angle of elevation of the top of the tower from the eye of the observer.



Sol. Height of tower (TW) = 22 m [Given]

Height of observer (AB) = 1.5 m [Given]

Distance between foot of tower and observer (BW) = 20.5 m [Given]

Let $\theta = \angle$ of elevation of the observer at the top of the tower

Now, $\quad TM = 22 \text{ m} - 1.5 \text{ m} = 20.5 \text{ m}$

$\quad AM = 20.5 \text{ m}$

$\therefore \quad \tan\theta = \dfrac{20.5}{20.5} = 1$

$\Rightarrow \quad \tan\theta = 1$

$\Rightarrow \quad \tan\theta = \tan 45°$

$\Rightarrow \quad \theta = 45°$

Hence, the angle of elevation of the top of the tower from observer's eye is 45°.

Q15. Show that $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$.

Sol. $\tan^2\theta$ or $\tan^4\theta$ can be converted into $\sec^2\theta$

So, $\quad \text{LHS} = \tan^4\theta + \tan^2\theta$

$= \tan^2\theta\,(\tan^2\theta + 1)$

$= (\sec^2\theta - 1)\cdot\sec^2\theta$

$[\because \quad \tan^2\theta = \sec^2\theta - 1 \text{ and } \tan^2\theta + 1 = \sec^2\theta]$

$= \sec^4\theta - \sec^2\theta$

$=$ RHS

Hence, proved.

EXERCISE 8.4

Q1. If cosec θ + cot θ = p, then prove that $\cos\theta = \dfrac{p^2-1}{p^2+1}$.

Sol. $\operatorname{cosec}\theta + \cot\theta = p$ [Given]

$$\Rightarrow \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = p$$

$$\Rightarrow \frac{1+\cos\theta}{\sin\theta} = p$$

$$\Rightarrow \frac{(1+\cos\theta)^2}{\sin^2\theta} = p^2$$

$$\Rightarrow \frac{(1+\cos\theta)^2}{(1-\cos^2\theta)} = p^2$$

$$\Rightarrow \frac{(1+\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = p^2$$

$$\Rightarrow \frac{(1+\cos\theta)}{(1-\cos\theta)} = \frac{p^2}{1}$$

$$\left[\begin{array}{l}\text{By using the rule of componendo and dividendo } \dfrac{a}{b} = \dfrac{c}{d} \\ \text{can be written as } \dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}\end{array}\right]$$

So, by using componendo and dividendo, we have

$$\frac{(1+\cos\theta)+(1-\cos\theta)}{(1+\cos\theta)-(1-\cos\theta)} = \frac{p^2+1}{p^2-1}$$

$$\Rightarrow \frac{2}{2\cos\theta} = \frac{p^2+1}{p^2-1} \quad \left(\text{By invertendo } \frac{a}{b} = \frac{c}{d}; \quad \frac{b}{a} = \frac{d}{c}\right)$$

$$\Rightarrow \cos\theta = \frac{p^2-1}{p^2+1}$$

Hence, proved.

Q2. Prove that $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$.

Sol. $\text{LHS} = \sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}$

$$= \sqrt{\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}} = \sqrt{\frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta}}$$

$$= \sqrt{\frac{1}{\sin^2\theta \cdot \cos^2\theta}} = \frac{1}{\sin\theta\cos\theta}$$

$$= \text{cosec}\,\theta\sec\theta$$

$$\text{RHS} = \tan\theta + \cot\theta$$

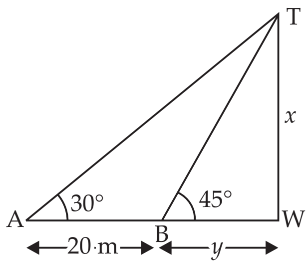
$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta \cdot \cos\theta}$$

$$= \text{cosec}\,\theta \cdot \sec\theta = \text{LHS}$$

Hence, proved.

Q3. The angle of elevation of the top of a tower from a certain point is 30°. If the observer moves 20 m towards the tower, the angle of elevation of the top increases by 15°. Find the height of the tower.



Sol. Consider the height of the vertical tower (TW) = x m (let)

Ist position of observer at A makes angle of elevation at the top of tower is 30°.

Now, observer moves towards the tower at new position B such that AB = 20 m. Let BW = y.

Now, angle of elevation of the top of tower is increased by 15° *i.e.*, it becomes 30° + 15° = 45°.

In ΔTWB, we have

$$\tan 45° = \frac{x}{y}$$

$$\Rightarrow \quad 1 = \frac{x}{y} \Rightarrow x = y \qquad \text{(I)}$$

Now, ΔTWA, we have

$$\tan 30° = \frac{x}{20 + y}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{x}{20 + x} \qquad \text{[From (I)]}$$

$$\Rightarrow \quad \sqrt{3}x = 20 + x$$

$$\Rightarrow \quad \sqrt{3}x - x = 20$$

$$\Rightarrow \quad x(\sqrt{3} - 1) = 20$$

$$\Rightarrow \qquad x = \frac{20}{(\sqrt{3}-1)} \times \frac{\sqrt{3} \quad 1}{(\sqrt{3}+1)}$$

$$\Rightarrow \qquad x = \frac{20(\sqrt{3}+1)}{3-1} = \frac{20(\sqrt{3}+1)}{2}$$

$$\Rightarrow \qquad x = 10(1.732 + 1)$$

$$\Rightarrow \qquad x = 10 \times 2.732 = 27.32 \text{ m}$$

Hence, the height of the tower is 27.32 m.

Q4. If $1 + \sin^2\theta = 3 \sin\theta \cos\theta$, then prove that $\tan\theta = 1$, or $\frac{1}{2}$.

Sol. To solve an equation in θ, we have to change it into one trigonometric ratio.

Given trigonometric equation is

$$1 + \sin^2\theta = 3\sin\theta\cos\theta$$

$$\Rightarrow \qquad \frac{1}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{3\sin\theta\cos\theta}{\sin^2\theta}$$

[Dividing by $\sin^2\theta$ both sides]

$$\Rightarrow \qquad \operatorname{cosec}^2\theta + 1 = 3\cot\theta$$

$$\Rightarrow \qquad 1 + \cot^2\theta + 1 - 3\cot\theta = 0$$

$$\Rightarrow \qquad \cot^2\theta - 3\cot\theta + 2 = 0$$

$$\Rightarrow \qquad \cot^2\theta - 2\cot\theta - 1\cot\theta + 2 = 0$$

$$\Rightarrow \qquad \cot\theta(\cot\theta - 2) - 1(\cot\theta - 2) = 0$$

$$\Rightarrow \qquad (\cot\theta - 2)(\cot\theta - 1) = 0$$

$$\Rightarrow \qquad \cot\theta - 2 = 0 \quad \text{or} \quad (\cot\theta - 1) = 0$$

$$\Rightarrow \qquad \cot\theta = 2 \quad \text{or} \quad \cot\theta = 1$$

$$\Rightarrow \qquad \tan\theta = \frac{1}{2} \quad \text{or} \quad \tan\theta = 1$$

Hence, $\tan\theta = \frac{1}{2}$, or 1.

Q5. Given that $\sin\theta + 2\cos\theta = 1$, then prove that $2\sin\theta - \cos\theta = 2$.

Sol. $\sin\theta + 2\cos\theta = 1$ [Given]

On squaring both sides, we get

$$(\sin\theta)^2 + (2\cos\theta)^2 + 2(\sin\theta)(2\cos\theta) = 1$$

$$\Rightarrow \qquad \sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta = 1$$

$$\Rightarrow \qquad 1 - \cos^2\theta + 4(1 - \sin^2\theta) + 4\sin\theta\cos\theta = 1$$

$$\Rightarrow \qquad 1 - \cos^2\theta + 4 - 4\sin^2\theta + 4\sin\theta\cos\theta = 1$$

$$\Rightarrow \qquad -\cos^2\theta - 4\sin^2\theta + 4\sin\theta\cos\theta = -4$$

$$\Rightarrow \qquad \cos^2\theta + 4\sin^2\theta - 4\sin\theta\cos\theta = 4$$

$$\Rightarrow \qquad (\cos\theta)^2 + (2\sin\theta)^2 - 2(\cos\theta)(2\sin\theta) = 4$$

$$\Rightarrow \qquad (2\sin\theta - \cos\theta)^2 = 2^2$$

Taking square root both sides, we have

$$2 \sin\theta - \cos\theta = 2$$

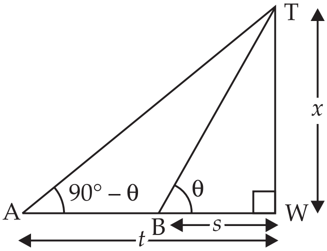
Hence, proved.

Q6. The angle of elevation of the top of a tower from two points distant s and t from its foot are complementary. Prove that the height of tower is $\sqrt{st}$.

Sol. Let the height of the vertical tower (TW) = x m

Points of observation A and B are at distances 't' and 's' from the foot of tower.

The angles of elevation of top of the tower from observation points A and B are $(90° - \theta)$ and θ which are complementary.



In ΔTWB, we have

$$\tan\theta = \frac{x}{s} \qquad \text{(I)}$$

Now, in ΔTWA, we have

$$\tan(90° - \theta) = \frac{x}{t}$$

$$\Rightarrow \quad \cot\theta = \frac{x}{t}$$

$$\Rightarrow \quad \cot\theta \cdot \tan\theta = \frac{x}{t} \cdot \frac{x}{s} \qquad \text{[Multiply by (I)]}$$

$$\Rightarrow \quad \frac{1}{\tan\theta} \cdot \tan\theta = \frac{x^2}{st}$$

$$\Rightarrow \quad \frac{x^2}{st} = 1$$

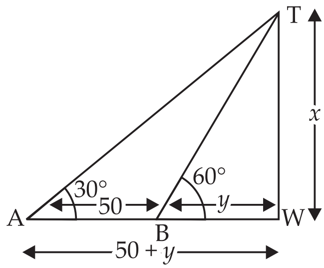
$$\Rightarrow \quad x^2 = st$$

$$\Rightarrow \quad x = \sqrt{st}$$

Hence, proved.

Q7. The shadow of a tower standing on a level plane is found to be 50 m. longer when Sun's elevation is 30° than when it is 60°. Find the height of the tower.

Sol. Let a tower TW of light x (let) is standing vertically upright on a level plane ABW. A and B are two positions of observation when angle of elevation changes from 30° to 60° respectively.



Let $\quad BW = y$

$AB = 50$ m $\qquad$ [Given]

In ΔTWB, we have

$$\tan 60° = \frac{x}{y}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x}{y}$$

$$\Rightarrow \quad x = \sqrt{3}y \qquad \text{(I)}$$

Now, in ΔTWA, we have

$$\tan 30° = \frac{x}{y + 50} \qquad \text{(II)}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{\sqrt{3}y}{y + 50} \qquad \text{[From (I)]}$$

$$\Rightarrow \quad 3y = y + 50$$

$$\Rightarrow \quad 3y - y = 50$$

$$\Rightarrow \quad 2y = 50$$

$$\Rightarrow \quad y = 25 \qquad \text{(I)}$$

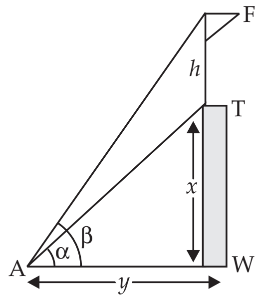
Now, $x = \sqrt{3}y$

$$\Rightarrow \quad x = \sqrt{3} \times 25$$

$$\Rightarrow \quad x = 25\sqrt{3} \text{ m}$$

Q8. A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height 'h'. At a point on the plane, the angles of elevation of the bottom and top of the flag staff are α and β, respectively. Prove that the height of the tower is $\left(\frac{h \tan \alpha}{\tan \beta - \tan \alpha}\right)$.

Sol. Let the height of vertical tower (TW) = x.
And, the height of flag staff (TF) = h (Given)
The angle of elevation at A on ground from the base and top of flag staff are α, β respectively.



Let AW = y

In ΔTWA, we have

$$\tan \alpha = \frac{x}{y}$$

$$\Rightarrow \quad y = \frac{x}{\tan \alpha} \qquad \text{(I)}$$

Now, in ΔFWA, we have

$$\tan \beta = \frac{x + h}{y}$$

$$\Rightarrow \quad y \tan \beta = x + h$$

$$\Rightarrow \quad \frac{x \tan \beta}{\tan \alpha} = x + h \qquad \text{[From (I)]}$$

$$\Rightarrow \quad x \tan \beta = x \tan \alpha + h \tan \alpha$$

$$\Rightarrow \quad x(\tan \beta - \tan \alpha) = h \tan \alpha$$

$$\Rightarrow \quad x = \frac{h \tan \alpha}{\tan \beta - \tan \alpha}$$

Hence, proved.

Q9. If $\tan\theta + \sec\theta = l$, then prove that $\sec\theta = \dfrac{l^2+1}{2l}$.

Sol. [Recall identity $\sec^2\theta - \tan^2\theta = 1$
and now change $\sec\theta + \tan\theta$ to $\sec^2\theta - \tan^2\theta$ by multiplying and dividing the given expression to $(\sec\theta - \tan\theta)$.

$$\sec\theta + \tan\theta = l \qquad \text{[Given]} \quad \text{(I)}$$

$$\Rightarrow \quad (\sec\theta + \tan\theta)\frac{(\sec\theta - \tan\theta)}{\sec\theta - \tan\theta} = l$$

$$\Rightarrow \quad \frac{\sec^2\theta - \tan^2\theta}{\sec\theta - \tan\theta} = l \qquad [\because 1 + \tan^2\theta = \sec^2\theta]$$

$$\Rightarrow \quad \frac{1}{\sec\theta - \tan\theta} = l$$

$$\text{or} \quad \sec\theta - \tan\theta = \frac{1}{l} \qquad \text{(II)}$$

Now, get $\sec\theta$ by eliminating $\tan\theta$ from (I) and (II).
It can be obtained by adding (I) and (II).

$$\Rightarrow \quad 2\sec\theta = l + \frac{1}{l}$$

$$\Rightarrow \quad 2\sec\theta = \frac{l^2+1}{l}$$

$$\Rightarrow \quad \sec\theta = \frac{l^2+1}{2l}$$

Hence, proved.

Q10. If $\sin\theta + \cos\theta = p$ and $\sec\theta + \operatorname{cosec}\theta = q$, then prove that $q(p^2 - 1) = 2p$.

Sol.
$$\sin\theta + \cos\theta = p \qquad \text{(I)}$$
$$\sec\theta + \operatorname{cosec}\theta = q \qquad \text{(II)}$$

[IInd expression can be changed into $\sin\theta$, $\cos\theta$ and eliminate trigonometric ratio from (I) and (II)]

$$\sec\theta + \operatorname{cosec}\theta = q$$

$$\Rightarrow \quad \frac{1}{\cos\theta} + \frac{1}{\sin\theta} = q$$

$$\Rightarrow \quad \frac{\sin\theta + \cos\theta}{\sin\theta\cos\theta} = \frac{q}{1}$$

$$\Rightarrow \quad \frac{p}{\sin\theta\cos\theta} = q \qquad \text{[Using (I)]}$$

$$\Rightarrow \quad \sin\theta\cos\theta = \frac{p}{q} \qquad \text{(III)}$$

$$\sin\theta + \cos\theta = p \qquad \text{[From (I)]}$$

$$\Rightarrow \quad (\sin\theta + \cos\theta)^2 = p^2 \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = p^2$$

$$\Rightarrow \quad 1 + 2\cdot\frac{p}{q} = p^2$$

$[\because \sin^2\theta + \cos^2\theta = 1$ and using (III)]

$$\Rightarrow \quad q + 2p = p^2 q$$

$$\Rightarrow \quad 2p = p^2 q - q$$

$$\Rightarrow \quad 2p = q(p^2 - 1)$$

Hence, proved.

Q11. If $a\sin\theta + b\cos\theta = c$, then prove that

$$a\cos\theta - b\sin\theta = \sqrt{a^2 + b^2 - c^2}.$$

Sol. $a\sin\theta + b\cos\theta = c$ [Given]

On squaring both sides, we get

$$(a\sin\theta)^2 + (b\cos\theta)^2 + 2(a\sin\theta)(b\cos\theta) = c^2$$

$$\Rightarrow \quad a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = c^2$$

$$\Rightarrow \quad a^2(1 - \cos^2\theta) + b^2(1 - \sin^2\theta) + 2ab\sin\theta\cos\theta = c^2$$

$$\Rightarrow \quad a^2 - a^2\cos^2\theta + b^2 - b^2\sin^2\theta + 2ab\sin\theta\cos\theta = c^2$$

$$\Rightarrow \quad -a^2\cos^2\theta - b^2\sin^2\theta + 2ab\sin\theta\cos\theta = c^2 - a^2 - b^2$$

$$\Rightarrow \quad a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = a^2 + b^2 - c^2$$

$$\Rightarrow \quad (a\cos\theta)^2 + (b\sin\theta)^2 - 2(a\cos\theta)(b\sin\theta) = a^2 + b^2 - c^2$$

$$\Rightarrow \quad (a\cos\theta - b\sin\theta)^2 = a^2 + b^2 - c^2$$

$$\Rightarrow \quad a\cos\theta - b\sin\theta = \pm\sqrt{a^2 + b^2 + c^2}$$

(Taking square root both sides)

Hence, $a\cos\theta - b\sin\theta = \sqrt{a^2 + b^2 - c^2}$

Hence, proved.

Q12. Prove that $\dfrac{1 + \sec\theta - \tan\theta}{1 + \sec\theta + \tan\theta} = \dfrac{1 - \sin\theta}{\cos\theta}$

Sol. Recall identity $\sec^2\theta - \tan^2\theta = 1$

$$\text{LHS} = \frac{1 + \sec\theta - \tan\theta}{1 + \sec\theta + \tan\theta}$$

$$= \frac{\sec^2\theta - \tan^2\theta + \sec\theta - \tan\theta}{1 + \sec\theta + \tan\theta}$$

$$= \frac{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta) + (\sec\theta - \tan\theta)}{1 + \sec\theta + \tan\theta}$$

$[\because a^2 - b^2 = (a - b)(a + b)]$

$$= \frac{(\sec\theta - \tan\theta)[\sec\theta + \tan\theta + 1]}{(\sec\theta + \tan\theta + 1)}$$

$$= \sec\theta - \tan\theta$$

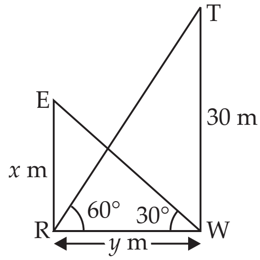
$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1 - \sin\theta}{\cos\theta} = \text{RHS}$$

Hence, proved.

Q13. The angle of elevation of the top of a tower 30 m high from the foot of another tower in the same plane is 60°, and the angle of elevation of the top of second tower from the foot of first tower is 30°. Find the distance between the two towers and also the height of the other tower.

Sol. Two vertical towers TW = 30 m and ER = x m (let) are standing on a horizontal plane RW = y(let). The angle of elevation from R to top of 30 m high tower is 60° and the angle of elevation of second tower from W is 30°.



In ΔERW,

$$\tan 30° = \frac{x}{y}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{x}{y}$$

$$\Rightarrow \quad y = \sqrt{3}x \qquad \text{(I)}$$

Now, In ΔTWR,

$$\tan 60° = \frac{30°}{y} \qquad \text{(I)}$$

$$\Rightarrow \quad \sqrt{3} = \frac{30}{\sqrt{3}x} \qquad \text{[From (I)]}$$

$$\Rightarrow \quad 3x = 30$$

$$\Rightarrow \quad x = 10 \text{ m}$$

Now, $y = \sqrt{3}x$

$$\Rightarrow \quad y = \sqrt{3}(10)$$

$$\Rightarrow \quad y = 1.732 \times 10$$

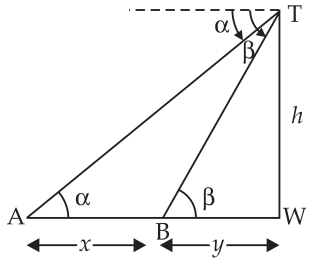
$$\Rightarrow \quad y = 17.32 \text{ m}$$

Hence, the distance between the two towers is 17.32 m and the height of the second tower is 10 m.

Q14. From the top of a tower h m high, the angles of depression of two objects, which are in line with the foot of the tower are α and β, ($\beta > \alpha$). Find the distance between the two objects.

Sol. Consider a vertical tower TW = h m. Two objects A and B are x m apart in the line joining A, B, and W and BW = y (let).

The angle of depression from the top a tower to objects A and B are α, and β respectively.



In ΔTWB, we have

$$\tan\beta = \frac{h}{y}$$

$$\Rightarrow \quad y = \frac{h}{\tan\beta} \qquad \text{(I)}$$

Now, in ΔTWA,

$$\tan\alpha = \frac{h}{x+y}$$

$$\Rightarrow \quad \tan\alpha\,(x+y) = h$$

$$\Rightarrow \quad \left(x + \frac{h}{\tan\beta}\right)\tan\alpha = h \qquad \text{[From(I)]}$$

$$\Rightarrow \quad x\tan\alpha + \frac{h\tan\alpha}{\tan\beta} = h$$

$$\Rightarrow \quad x\tan\alpha = h - \frac{h\tan\alpha}{\tan\beta}$$

$$\Rightarrow \quad x\tan\alpha = \frac{h\tan\beta - h\tan\alpha}{\tan\beta}$$

$$\Rightarrow \quad x = \frac{h[\tan\beta - \tan\alpha]}{\tan\alpha\cdot\tan\beta}$$

$$\Rightarrow \quad x = h\left[\frac{\tan\beta}{\tan\alpha\cdot\tan\beta} - \frac{\tan\alpha}{\tan\alpha\cdot\tan\beta}\right]$$

$$\Rightarrow \quad x = h\left[\frac{1}{\tan\alpha} - \frac{1}{\tan\beta}\right]$$

$$\Rightarrow \quad x = h[\cot\alpha - \cot\beta]$$

Hence, the distance between the two objects is $h(\cot\alpha - \cot\beta)$ m.

Q15. A ladder rests against a vertical wall at an inclination α to the horizontal. Its foot is pulled away from the wall through a distance p, so that its upper end slides a distance q down the wall and then the ladder makes an angle β with horizontal. Show that $\frac{p}{q} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta}$.

Sol. Consider a vertical wall WB. Two positions AW and LD of a ladder as shown in figure such that LA = p, WD = q and LD = AW = z. Angle of inclination of ladder at two positions A and L are α and β respectively. Let AB = y and DB = x.

In ΔABW, we have

$$\sin\alpha = \frac{x+q}{z}$$

and $$\cos\alpha = \frac{y}{z}$$

In ΔLBD, we have

$$\sin\beta = \frac{x}{z}$$

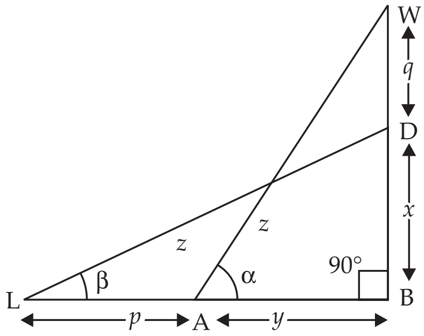
and $$\cos\beta = \frac{y+p}{z}$$

Taking RHS = $\dfrac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta}$

$$= \frac{\frac{y+p}{z} - \frac{y}{z}}{\frac{x+q}{z} - \frac{x}{z}} = \frac{\frac{y+p-y}{z}}{\frac{x+q-x}{z}}$$

$$= \frac{p}{z} \div \frac{q}{z} = \frac{p}{z} \times \frac{z}{q}$$

$$= \frac{p}{q} = \text{LHS}$$



Hence, proved.

Q16. The angle of elevation of the top of a vertical tower from a point on the ground is 60°. From another point 10 m vertically above the first, its angle of elevation is 45°. Find the height of the tower.

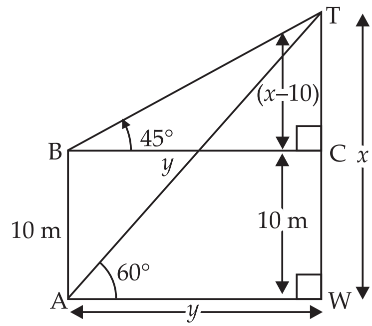
Sol. Let the height of the vertical tower TW = x m.

It stands on a horizontal plane AW = y.

Also BC = y.

Observation point B is 10 m above the first observation point A.

The angles of elevation from point of observations A and B are 60° and 45° respectively.



$$\text{TC} = x - 10$$

In right angled triangle TBC, we have

$$\tan 45° = \frac{x-10}{y}$$

$$\Rightarrow \quad 1 = \frac{x-10}{y}$$

$$\Rightarrow \quad y = x - 10 \qquad \text{(I)}$$

Now, in ΔTAW,

$$\tan 60° = \frac{x}{y}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x}{x-10} \qquad \text{[From (I)]}$$

$$\Rightarrow \quad \sqrt{3}x - 10\sqrt{3} = x$$

$$\Rightarrow \quad \sqrt{3}x - x = 10\sqrt{3}$$

$$\Rightarrow \quad x(\sqrt{3} - 1) = 10\sqrt{3}$$

$$\Rightarrow \quad x = \frac{10\sqrt{3}}{(\sqrt{3}-1)} \times \frac{(\sqrt{3}+1)}{\sqrt{3}+1}$$

$$\Rightarrow \quad x = \frac{10(3+\sqrt{3})}{(\sqrt{3})^2 - 1}$$

$$= \frac{10 \times (3 + 1.732)}{3 - 1} = \frac{10 \times 4.732}{2} = 10 \times 2.366$$

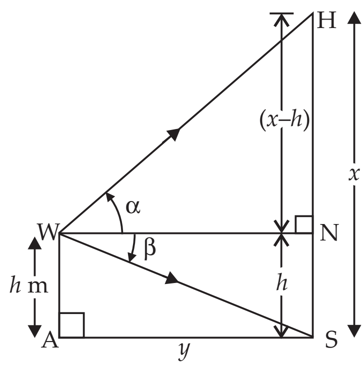
$$\Rightarrow \quad x = 23.66 \text{ m}$$

Hence, the height of the tower = 23.66 m.

Q17. A window of a house is h m above the ground. From the window, the angles of elevation and depression of the top and the bottom of another house situated on the opposite side of the lane are found to be α and β respectively. Prove that the height of the other house is h (1 + tan α cot β) m.

Sol. Window W, h m above the ground point A, another house HS = x(m), AS = y m away from observation window, AS = WN = y (let), NS = h, HN = $(x - h)$.

Angle of elevation and depression of top and bottom of house HS from window W are α, β respectively.



In right angled ΔWNS,

$$\tan \beta = \frac{h}{y}$$

$$\Rightarrow \quad y = \frac{h}{\tan \beta} \qquad \text{(I)}$$

Now, in right angled ΔHNW,

$$\Rightarrow \quad \tan \alpha = \frac{x - h}{y}$$

$$\Rightarrow \quad y \tan \alpha = x - \text{m } h$$

$$\Rightarrow \quad \frac{h \tan \alpha}{\tan \beta} = x - h \quad \text{[From (I)]}$$

$$\Rightarrow \quad h \tan \alpha = x \tan \beta - h \tan \beta$$

$$\Rightarrow \quad x \tan \beta = h \tan \alpha + h \tan \beta$$

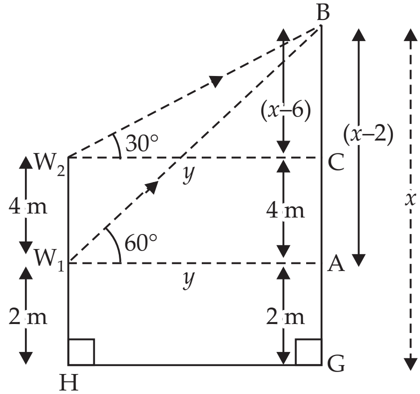
$$\Rightarrow \quad x = \frac{h(\tan \alpha + \tan \beta)}{\tan \beta}$$

$$\Rightarrow \quad x = h\left[\frac{\tan \alpha}{\tan \beta} + \frac{\tan \beta}{\tan \beta}\right]$$

$$\Rightarrow \quad x = h\,[\tan \alpha \cdot \cot \beta + 1]$$

Hence, the height of the house on the other side of the observer is h [1 + tan α . cot β] m.

Q18. The lower window of a house is at a height of 2 m above the ground and its upper window is 4 m vertically above the lower window. At certain instant the angles of elevation of a balloon from these windows are observed to be 60° and 30°, respectively. Find the height of the balloon above the ground.



Sol. Let B be a balloon at a height GB = x m.

Let W_1 be the window, which is 2 m above the ground H.

$\therefore \quad W_1H = 2$ m

$\Rightarrow \quad AG = 2$ m

Let W_2 be the second window, which is the 4 m above the window W_1.

$\therefore \quad W_2W_1 = AC = 4$ m

The angles of elevation of balloon B from W_1, W_2 are 60° and 30° respectively.

$$BA = (x - 2) \text{ m}$$

$$BC = x - 2 - 4 = (x - 6) \text{ m}$$

In right angled ΔW_2CB, we have

$$\tan 30° = \frac{x-6}{y}$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{x-6}{y}$$

$$\Rightarrow \quad y = \sqrt{3}(x-6) \qquad \text{(I)}$$

Now, in right angled ΔW_1AB,

$$\tan 60° = \frac{x-2}{y}$$

$$\Rightarrow \quad \sqrt{3} = \frac{x-2}{y}$$

$$\Rightarrow \quad \sqrt{3}y = (x-2) \qquad \text{(II)}$$

$$\Rightarrow \quad \sqrt{3}\cdot\sqrt{3}(x-6) = x-2 \qquad \text{[From (I)]}$$

$$\Rightarrow \quad 3x - 18 = x - 2$$

$$\Rightarrow \quad 3x - x = 18 - 2$$

$$\Rightarrow \quad 2x = 16$$

$$\Rightarrow \quad x = 8 \text{ m}$$

Hence, the height of the balloon above the ground is 8 m.