EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Give an example of a statement P(n) which is true for all $n \geq 4$ but P(1), P(2) and P(3) are not true. Justify your answer.

Sol. The required statement is $P(n) = 2n < n!$

Justification: $P(n)\ :\ 2n < n!$

$$P(1) : 2.1 < 1! \Rightarrow 2 < 1 \text{ not true}$$
$$P(2) : 2.2 < 2! \Rightarrow 4 < 2.1 \Rightarrow 4 < 2 \text{ not true}$$
$$P(3) : 2.3 < 3! \Rightarrow 6 < 3.2.1 \Rightarrow 6 < 6 \text{ not true}$$
$$P(4) : 2.4 < 4! \Rightarrow 8 < 4.3.2.1 \Rightarrow 8 < 24 \text{ True}$$
$$P(5) : 2.5 < 5! \Rightarrow 10 < 5.4.3.2.1 \Rightarrow 10 < 120 \text{ True}$$

Hence, $P(n) = 2n < n!$ is not true for P(1), P(2) and P(3) but it is true for all values of $n \geq 4$.

Q2. Give an example of a statement P(n) which is true for all n. Justify your answer.

Sol. The required statement is

$$P(n) : 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Justification: P(1) : $1 = \frac{1(1+1)}{2}$

P(k) : $1 + 2 + 3 + ... + k = \frac{k(k+1)}{2}$. Let it be true.

$$P(k+1) : 1 + 2 + 3 + ... + k + (k+1)$$
$$= \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{(k+1)(k+2)}{2}$$

Hence, P(k + 1) is true whenever P(k) is true.

Prove each of the statements in Exercise 3 to 6 by the principle of Mathematical Induction.

Q3. $4^n - 1$ is divisible by 3, for each natural number n.

Sol. Let $P(n) : 4^n - 1$

Step 1: $P(1) = 4 - 1 = 3$ which is divisible by 3, so it is true.

Step 2: $P(2) = 4^k - 1 = 3\lambda$. Let it be true.

Step 3: $P(k + 1) = 4^{k+1} - 1$

$= 4^k .4 - 1 = 4.4^k - 4 + 3 = 4(4^k - 1) + 3$

$= 4 (3\lambda) + 3$ (from Step 2)

$= 3[4\lambda + 1]$ which is true as it is divisible by 3.

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

Q4. $2^{3n} - 1$ is divisible by 7, for all natural numbers n.

Sol. Let $P(n) : 2^{3n} - 1$

Step 1: $P(1) = 2^{3.1} - 1 = 8 - 1 = 7$ which is divisible by 7.

So, P(1) is true.

Step 2: $P(k) = 2^{3k} - 1 = 7\lambda$. Let it be true.

Step 3: $P(k + 1) = 2^{3(k+1)} - 1$

$= 2^{3k+3} - 1 = 2^3.2^{3k} - 8 + 7 = 8.2^{3k} - 8 + 7$

$= 8(2^{3k} - 1) + 7$ (from Step 2)

$= 8.7\lambda + 7$

$= 7(8\lambda + 1)$ which is true as it is divisible by 7

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

Q5. $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.

Sol. Let $P(n) : n^3 - 7n + 3$

Step 1: $P(1) = (1)^3 - 7(1) + 3$

$= 1 - 7 + 3 = -3$ which is divisible by 3.

So, it is true for P(1).

Step 2: $P(k) : k^3 - 7k + 3 = 3\lambda$. Let it be true.

$\Rightarrow k^3 = 3\lambda + 7k - 3$

Step 3: $P(k + 1) = (k + 1)^3 - 7(k + 1) + 3$

$= k^3 + 1 + 3k^2 + 3k - 7k - 7 + 3$

$= k^3 + 3k^2 - 4k - 3$

$= (3\lambda + 7k - 3) + 3k^2 - 4k - 3$ (from Step 2)

$= 3k^2 + 3k + 3\lambda - 6$

$= 3(k^2 + k + \lambda - 2)$ which is divisible by 3.

So it is true for $P(k + 1)$.

Hence, $P(k + 1)$ is true whenever it is true for $P(k)$.

Q6. $3^{2n} - 1$ is divisible by 8, for all natural numbers n.

Sol. Let $P(n) : 3^{2n} - 1$

Step 1: $P(1) : 3^2 - 1 = 9 - 1 = 8$ which is divisible by 8.

So, it is true for P(1).

Step 2: $P(k) = 3^{2k} - 1 = 8\lambda$. Let it be true.

Step 3: $P(k + 1) = 3^{2(k+1)} - 1$

$= 3^{2k+2} - 1 = 3^2.3^{2k} - 9 + 8 = 9(3^{2k} - 1) + 8$

$= 9.8\lambda + 8$ (from Step 2)

$= 8[9\lambda + 1]$ which is divisible by 8.

So it is true for $P(k + 1)$.

Hence, $P(k + 1)$ is true whenever it is true for $P(k)$.

Q7. For any natural number n, $7^n - 2^n$ is divisible by 5.

Sol. Let $P(n) : 7^n - 2^n$

Step 1: $P(1) : 7^1 - 2^1 = 5$ which is divisible by 5.
So it is true for P(1).

Step 2: $P(k) : 7^k - 2^k = 5\lambda$. Let it be true for $P(k)$.

Step 3:
$$\begin{aligned} P(k+1) &= 7^{k+1} - 2^{k+1} \\ &= 7^{k+1} + 7^k.2 - 7^k.2 - 2^{k+1} \\ &= (7^{k+1} - 7^k.2) + 7^k.2 - 2^{k+1}) \\ &= 7^k(7-2) + 2.(7^k - 2^k) \\ &= 5.7^k + 2.5\lambda \qquad \text{(from Step 2)} \\ &= 5(7^k + 2\lambda) \text{ which is divisible by 5.} \end{aligned}$$

So, it is true for $P(k+1)$.
Hence, $P(k+1)$ is true whenever $P(k)$ is true.

Q8. For any natural number n, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

Sol. Let $P(n) : x^n - y^n$

Step 1: $P(1) : x^1 - y^1 = x - y$ which is divisible by $x - y$.
So P(1) is true.

Step 2: $P(k) : x^k - y^k = (x - y)\lambda$. Let it be true.

Step 3:
$$\begin{aligned} P(k+1) &= x^{k+1} - y^{k+1} = x^{k+1} - x^k y - x^k y - y^{k+1} \\ &= (x^{k+1} - x^k y) + (x^k y - y^{k+1}) \\ &= x^k(x-y) + y(x^k - y^k) \\ &= x^k (x-y) + y.(x-y)\lambda \qquad \text{(from Step 2)} \\ &= (x-y)(x^k + y\lambda) \text{which is divisible by } (x-y). \end{aligned}$$

So, it is true for $P(k+1)$.

Q9. $n^3 - n$ is divisible by 6, for each natural number $n \geq 2$.

Sol. Let $P(n) : n^3 - n$

Step 1: $P(2) : 2^3 - 2 = 6$ which is divisible by 6. So it is true for P(2).

Step 2: $P(k) : k^3 - k = 6\lambda$. Let it be true for $k \geq 2$

$$\Rightarrow \qquad k^3 = 6\lambda + k \qquad \ldots(i)$$

Step 3:
$$\begin{aligned} P(k+1) &= (k+1)^3 - (k+1) \\ &= k^3 + 1 + 3k^2 + 3k - k - 1 \\ &= k^3 - k + 3(k^2 + k) \\ &= 6\lambda + 3(k^2 + k) \qquad \text{[from } (i)] \end{aligned}$$

We know that $3(k^2 + k)$ is divisible by 6 for every value of $k \in$ N.
Hence $P(k+1)$ is true whenever $P(k)$ is true.

Q10. $n(n^2 + 5)$ is divisible by 6, for each natural number n.

Sol. Let $P(n) : n(n^2 + 5)$

Step 1: $P(1) : 1(1 + 5) = 6$ which is divisible by 6. So it is true for P(1).

Step 2: $P(k) : \; k(k^2 + 5) = 6\lambda$. Let it be true.

$\Rightarrow \quad k^3 + 5k = 6\lambda$

$\Rightarrow \quad k^3 = 6\lambda - 5k \quad ...(i)$

Step 3: $P(k+1) = (k+1)[(k+1)^2 + 5]$

$= (k+1)[k^2 + 1 + 2k + 5]$

$= (k+1)[k^2 + 2k + 6]$

$= k^3 + 2k^2 + 6k + k^2 + 2k + 6$

$= k^3 + 3k^2 + 8k + 6$

$= k^3 + 5k + 3k^2 + 3k + 6$

$= 6\lambda - 5k + 5k + 3(k^2 + k + 2) \quad$ [From (i)]

$= 6\lambda + 3(k^2 + k + 2)$

We know that $k^2 + k + 2$ is divisible by 2 for each value of $k \in$ N, so, let $k^2 + k + 2 = 2m$.

So $\quad P(k+1) = 6\lambda + 3.2m = 6(\lambda + m)$ which is divisible by 6.

Hence, P(k + 1) is true whenever P(k) is true.

Q11. $n^2 < 2^n$, for all natural numbers $n \geq 5$.

Sol. Let P(n) : $n^2 < 2^n$ for all natural numbers, $n \geq 5$

Step 1: P(5) : $1^5 < 2^5 \Rightarrow 1 < 32$ which true for P(5).

Step 2: P(k) : $k^2 < 2^k$. Let it be true for $k \in$ N.

Step 3: P(k + 1) : $\quad (k+1)^2 < 2^{k+1}$

From Step 2, we get

$k^2 < 2^k$

$\Rightarrow \quad k^2 + 2k + 1 < 2^k + 2k + 1$

$\Rightarrow \quad (k+1)^2 < 2^k + 2k + 1 \quad ...(i)$

Since $\quad (2k+1) < 2^k$

So $\quad k^2 + 2k + 1 < 2^k + 2^k$

$\Rightarrow \quad k^2 + 2k + 1 < 2.2^k$

$\Rightarrow \quad k^2 + 2k + 1 < 2^{k+1} \quad ...(ii)$

From eqn. (i) and (ii), we get $(k+1)^2 < 2^{k+1}$.

Hence, P(k + 1) is true whenever P(k) is true for $k \in$ N, $n \geq 5$.

Q12. $2n < (n+2)!$ for all natural number n.

Sol. Let P(n) : $2n < (n+2)!$ for all $k \in$ N.

Step 1: P(1) : $\quad 2.1 < (1+2)!$

$\Rightarrow \quad 2 < 3! \Rightarrow 2 < 6$ which is true for P(1)

$(\because 3! = 3 \times 2 \times 1 = 6)$

Step 2: P(k) : $\quad 2k < (k+2)!$. Let it be true for P(k)

Step 3: P(k + 1) : $\quad 2(k+1) < (k+1+2)!$

Since $\quad 2k < (k+2)! \quad$ (from Step 2)

$\Rightarrow \quad 2k + 2 < (k+2)! + 2$

$\Rightarrow \quad 2(k+1) < (k+2)! + 2$

Also, $\quad (k+2)! + 2 < (k+3)!$

$\therefore \quad 2(k+1) < (k+3)!$

$\Rightarrow \quad 2(k+1) < (k+2+1)!$ which is true for $P(k+1)$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

Q13. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$ for all natural numbers $n \geq 2$.

Sol. Let $P(n)$: $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}, \ \forall\, n \geq 2$

Step 1: $P(2)$: $\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$ which is true.

Step 2: $P(k)$: $\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}}$. Let it be true.

Step 3: $P(k+1)$: $\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k+1}}$

From Step 2, we have

$$\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}}$$

$$\Rightarrow \quad \sqrt{k} + \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

$$\Rightarrow \quad \frac{\sqrt{k}.\sqrt{k+1}+1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \qquad \dots(i)$$

Now if $\sqrt{k+1} < \frac{\sqrt{k}.\sqrt{k+1}+1}{\sqrt{k+1}}$

$$\Rightarrow \quad (k+1) < \sqrt{k}.\sqrt{k+1}+1$$

$$\Rightarrow \quad k < \sqrt{k}.\sqrt{k+1} \qquad \dots(ii)$$

From eqn. (*i*) and (*ii*) we get

$$\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k+1}}$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

Q14. $2 + 4 + 6 + \dots + 2n = n^2 + n$, for all natural numbers n.

Sol. Let $P(n)$: $2 + 4 + 6 + \dots + 2n = n^2 + n, \ \forall\, n \in N$

Step 1: $P(1)$: $2 = 1^2 + 1 = 2$

which is true for $P(1)$

Step 2: $P(k)$: $2 + 4 + 6 + \dots + 2k = k^2 + k$. Let it be true.

Step 3: $P(k+1)$: $2 + 4 + 6 + \dots + 2k + 2k + 2$

$$= k^2 + k + 2k + 2 = k^2 + 3k + 2$$

$$= k^2 + 2k + k + 1 + 1$$

$$= (k+1)^2 + (k+1)$$

Which is true for $P(k+1)$

So, $P(k+1)$ is true whenever $P(k)$ is true.

Q15. $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$ for all natural numbers n.

Sol. Let $P(n) : 1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$, $n \in N$.

$P(n) : 2^0 + 2^1 + 2^2 + ... + 2^n = 2^{n+1} - 1$

Step 1: $P(1) = 2^0 = 2^{0+1} - 1 = 2 - 1 = 1 = 2^0$ which is true.

Step 2: $P(k) = 2^0 + 2^1 + 2^2 + ... + 2^k = 2^{k+1} - 1$. Let it be true.

Step 3: $P(k + 1) = 2^0 + 2^1 + 2^2 + ... + 2^k + 2^{k+1}$.

$= 2^{k+1} - 1 + 2^{k+1} = 2.2^{k+1} - 1 = 2^{k+2} - 1$

$= 2^{(k+1)+1} - 1$ which is true for $P(k + 1)$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

Q16. $1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$, for all natural numbers n.

Sol. Let $P(n) : 1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$, $\forall n \in N$

Step 1: $P(1)$: $1 = 1(2.1 - 1) = 1$ which is true for $P(1)$

Step 2: $P(k)$: $1 + 5 + 9 + ... + (4k - 3) = k(2k - 1)$. Let it be true.

Step 3: $P(k + 1) : 1 + 5 + 9 + ... + (4k - 3) + (4k + 1)$

$= k(2k - 1) + (4k + 1) = 2k^2 - k + 4k + 1$

$= 2k^2 + 3k + 1 = 2k^2 + 2k + k + 1$

$= 2k(k + 1) + 1(k + 1) = (2k + 1)(k + 1)$

$= (k + 1)(2k + 2 - 1) = (k + 1) [2(k + 1) - 1]$

Which is true for $P(k + 1)$.

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

Q17. A sequence a_1, a_2, a_3 ... is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all natural numbers.

Sol. Given that: $a_1 = 3$

$a_2 = 7a_{2-1} = 7.a_1 = 7.3 = 21$

$a_3 = 7.a_{3-1} = 7.a_2 = 7.21 = 147$

...

Let $P(n) : a_n = 3.7^{n-1}$, $\forall n \in N$

Step 1: $P(2)$: $a_2 = 3.7^{2-1} = 21 \Rightarrow 21 = 21$ which is true for $P(2)$.

Step 2: $P(k)$: $a_k = 3.7^{k-1}$. Let it be true.

Step 3: $a_k = 7a_{k-1}$ (given)

Put $k = k + 1$

$a_{k+1} = 7a_k = 7(3.7^{k-1}) = 3.7^{k+1-1} = 3.7^{(k+1)-1}$

which is true for $P(k + 1)$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

Q18. A sequence b_0, b_1, b_2, ... is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all natural numbers k. Show that $b_n = 5 + 4n$ for all natural number n using Mathematical Induction.

Sol. We have $b_0 = 5$ and $b_k = 4 + b_{k-1}$

$\Rightarrow b_0 = 5, b_1 = 4 + b_0 = 4 + 5 = 9$ and $b_2 = 4 + b_1 = 4 + 9 = 13$

Let $P(n) : b_n = 5 + 4n$

Step 1: $P(1)$: $b_1 = 5 + 4 = 9 \Rightarrow 9 = 9$ which is true.

Step 2: P(k) : $b_k = 5 + 4k$. Let it be true $\forall\, k \in$ N

Step 3: Given that:

$$P(k) = 4 + b_{k-1}$$

$$\Rightarrow \quad P(k+1) = 4 + b_{k+1-1}$$

$$\Rightarrow \quad P(k+1) = 4 + b_k = 4 + 5 + 4k$$

$$\Rightarrow \quad P(k+1) = 5 + 4(k+1) \text{ which is true for } P(k+1)$$

Hence, P(k + 1) is true whenever P(k) is true.

Q19. A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \dfrac{d_{k-1}}{k}$ for all natural numbers $k \geq 2$. Show that $d_n = \dfrac{2}{n!}$ for all $n \in$ N.

Sol. Given that: $d_1 = 2$ and $d_k = \dfrac{d_{k-1}}{k}$

Let P(n) : $d_n = \dfrac{2}{n!}$

Step 1: P(1) : $d_1 = \dfrac{2}{1!} = 2$ which is true for P(1).

Step 2: P(k) : $d_k = \dfrac{2}{k!}$. Let it be true for P(k).

Step 3: Given that: $d_k = \dfrac{d_{k-1}}{k}$

$$\therefore \quad d_{k+1} = \frac{d_{k+1-1}}{k+1} = \frac{d_k}{k+1}$$

$$\Rightarrow \quad d_{k+1} = \frac{1}{k+1} . d_k = \frac{1}{k+1} . \frac{2}{k!}$$

$$\Rightarrow \quad d_{k+1} = \frac{2}{(k+1)!} \text{ Which is true for } P(k+1)$$

Hence, P(k + 1) is true whenever P(k) is true.

Q20. Prove that for all $n \in$ N.

$$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + (n-1)\beta)$$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right] \sin\left[\frac{n\beta}{2}\right]}{\sin \frac{\beta}{2}}$$

Sol. Let P(n) : $\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos[\alpha + (n-1)\beta]$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin \frac{n\beta}{2}\right]}{\sin \frac{\beta}{2}}$$

Step 1: P(1) : $\cos\alpha = \dfrac{(\cos\alpha)\left(\sin\dfrac{\beta}{2}\right)}{\sin\dfrac{\beta}{2}} = \cos\alpha$

which is true for P(1)

Step 2: P(k) : $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(k-1)\beta]$

$$= \frac{\cos\left[\alpha + \left(\dfrac{k-1}{2}\right)\beta\right]\sin\left(\dfrac{k\beta}{2}\right)}{\sin\dfrac{\beta}{2}}$$. Let it be true.

Step 3: P($k+1$) : $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos[\alpha+(k-1)\beta]$
$+ \cos[\alpha+(k+1-1)\beta]$

$$= \frac{\cos\left[\alpha + \left(\dfrac{k-1}{2}\right)\beta\right]\sin\left(\dfrac{k\beta}{2}\right)}{\sin\dfrac{\beta}{2}} + \cos(\alpha+k\beta) \qquad \text{(from Step 2)}$$

$$= \frac{2\cos\left[\alpha + \left(\dfrac{k-1}{2}\right)\beta\right]\sin\left(\dfrac{k\beta}{2}\right) + 2\cos(\alpha+k\beta).\sin\dfrac{\beta}{2}}{2\sin\dfrac{\beta}{2}}$$

$$= \frac{\sin\left[\alpha+k\beta-\dfrac{\beta}{2}\right] - \sin\left[\alpha-\dfrac{\beta}{2}\right] + \sin\left[\alpha+k\beta+\dfrac{\beta}{2}\right] - \sin\left[\alpha+k\beta-\dfrac{\beta}{2}\right]}{2\sin\dfrac{\beta}{2}}$$

$[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$

$$= \frac{\sin\left[\alpha+k\beta+\dfrac{\beta}{2}\right] - \sin\left(\alpha-\dfrac{\beta}{2}\right)}{2\sin\dfrac{\beta}{2}}$$

$$= \frac{2\cos\left(\alpha+\dfrac{k\beta}{2}\right)\sin(k+1)\dfrac{\beta}{2}}{2\sin\dfrac{\beta}{2}}$$

$\left[\because \sin A - \sin B = 2\cos\dfrac{A+B}{2}.\sin\dfrac{A-B}{2}\right]$

$$= \frac{\cos\left(\alpha+\dfrac{k\beta}{2}\right).\sin(k+1)\dfrac{\beta}{2}}{\sin\dfrac{\beta}{2}}$$

$$= \frac{\cos\left[\alpha + \left(\frac{k+1-1}{2}\right)\beta\right] \sin\left(\frac{k+1}{2}\right)\beta}{\sin\frac{\beta}{2}} \text{ which is true for } P(k+1)$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

Q21. Prove that: $\cos\theta.\cos 2\theta.\cos 2^2\theta \ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$, for all $n \in N$.

Sol. Let $P(n) : \cos\theta.\cos 2\theta.\cos 2^2\theta \ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}$, $\forall\, n \in N$.

Step 1: $P(1) : \cos\theta = \frac{\sin 2^1\theta}{2^1 \sin\theta} = \frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta$

$\Rightarrow$ $\cos\theta = \cos\theta$ which is true for $P(1)$

Step 2: $P(k) : \cos\theta.\cos 2\theta.\cos 2^2\theta \ldots \cos 2^{k-1}\theta = \frac{\sin 2^k\theta}{2^k \sin\theta}$

Let it be true for $P(k)$.

Step 3: $P(k+1) : \cos\theta.\cos 2\theta.\cos 2^2\theta \ldots \cos 2^{k-1}\theta \,.\, \cos 2^{(k+1)-1}\theta$

$$= \frac{\sin 2^k\theta}{2^k \sin\theta}.\cos 2^{(k+1)-1}\theta = \frac{\sin 2^k\theta}{2^k \sin\theta}.\cos 2^k\theta$$

$$= \frac{2\sin 2^k\theta.\cos 2^k\theta}{2.2^k \sin\theta}$$

$$= \frac{\sin 2.2^k\theta}{2^{k+1}\sin\theta} \qquad [\because 2\sin\theta\cos\theta = \sin 2\theta]$$

$$= \frac{\sin 2^{k+1}\theta}{2^{k+1}\sin\theta} \text{ which is true for } P(k+1).$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

Q22. Prove that $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\frac{n\theta}{2}.\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}, \text{ for all } n \in N.$$

Sol. Let $P(n) : \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\frac{n\theta}{2}.\sin\left(\frac{n+1}{2}\right)\theta}{\sin\frac{\theta}{2}}, n \in N.$$

Step 1: $P(1) : \sin\theta = \frac{\sin\frac{\theta}{2}.\sin\left(\frac{1+1}{2}\right)\theta}{\sin\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}.\sin\theta}{\sin\frac{\theta}{2}} = \sin\theta$

$\therefore$ $\sin\theta = \sin\theta$ which is true for P(1).

Step 2: P(k) : $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin k\theta$

$$= \frac{\sin\frac{k\theta}{2}.\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}}.$$ Let it be true for P(k).

Step 3: P(k + 1): $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin(k+1)\theta$

$$= \frac{\sin\frac{k\theta}{2}.\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}} + \sin(k+1)\theta$$

$$= \frac{\sin\frac{k\theta}{2}.\sin\left(\frac{k+1}{2}\right)\theta + \sin(k+1)\theta.\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{k\theta}{2}.\sin\left(\frac{k+1}{2}\right)\theta + 2\sin(k+1)\theta.\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\left(\frac{k\theta}{2} - \frac{k+1}{2}\theta\right) - \cos\left(\frac{k\theta}{2} + \frac{k+1}{2}\theta\right) + \cos\left[(k+1)\theta - \frac{\theta}{2}\right] - \cos\left[(k+1)\theta + \frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\left(-\frac{\theta}{2}\right) - \cos\left(k\theta + \frac{\theta}{2}\right) + \cos\left(k\theta + \frac{\theta}{2}\right) - \cos\left(k\theta + \frac{3\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\left(\frac{\theta}{2}\right) - \cos\left(k\theta + \frac{3\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{-2\sin\left(\frac{\frac{\theta}{2} + k\theta + \frac{3\theta}{2}}{2}\right).\sin\left(\frac{\frac{\theta}{2} - k\theta - \frac{3\theta}{2}}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$\left[\because \cos A - \cos B = -2\sin\frac{(A+B)}{2}\sin\frac{(A-B)}{2}\right]$$

$$= \frac{-2\sin\left(\frac{k\theta+2\theta}{2}\right).\sin\left(\frac{-k\theta-\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\frac{k\theta+2\theta}{2}\right).\sin\left(\frac{k\theta+\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left[\frac{(k+1)+1}{2}\right]\theta.\sin\left[\frac{k+1}{2}\right]\theta}{\sin\frac{\theta}{2}}$$ which is true for P(k + 1).

Hence, P(k + 1) is true whenever P(k) is true.

Q23. Show that: $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number for all $n \in$ N.

Sol. Let P(n) : $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$, $\forall\, n \in$ N.

Step 1: P(1) : $\frac{1^5}{5} + \frac{1^3}{3} + \frac{7.1}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1 \in N$

Which is true for P(1).

Step 2: P(k) : $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7.k}{15}$. Let it be true for P($k$) and let $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} = \lambda$.

Step 3: $P(k+1) = \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$

$$= \frac{1}{5}\left[k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1\right] + \frac{1}{3}[k^3 + 3k^2 + 3k + 1] + \frac{7}{15}k + \frac{7}{15}$$

$$= \left(\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}\right) + (k^4 + 2k^3 + 3k^2 + 2k) + \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$$

$= \lambda + k^4 + 2k^3 + 3k^2 + 2k + 1$ [from Step 2]

= positive integers = natural number

Which is true for P(k + 1).

Hence, P(k + 1) is true whenever P(k) is true.

Q24. Prove that: $\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{13}{24}$ for all natural numbers, $n > 1$.

Sol. Let P(n) : $\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{13}{24}$, $\forall\, n \in$ N

Step 1: P(2) : $\frac{1}{2+1}+\frac{1}{2+2}>\frac{13}{24} \Rightarrow \frac{1}{3}+\frac{1}{4}>\frac{13}{24}$

$\Rightarrow \frac{7}{12}>\frac{13}{24} \Rightarrow \frac{14}{24}>\frac{13}{24}$ which is true for P(2).

Step 2: P(k) : $\frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{2k}>\frac{13}{24}$. Let it be true for P($k$).

Step 3: P(k + 1) : $\frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}$

Since $\frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{2k}>\frac{13}{24}$

So $\frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}$

Which is true for P(k + 1).

Hence, P(k + 1) is true whenever P(k) is true.

Q25. Prove that number of subsets of a set containing n distinct elements is 2^n, for all $n \in$ N.

Sol. Let P(n) : Number of subsets of a set containing n distinct elements is 2^n, $\forall\, n \in$ N

Step 1: It is clear that P(1) is true for n = 1. Number of subsets = 2^1 = 2. Which is true.

Step 2: P(k) is assumed to be true for $n = k$. Since the number of subsets = 2^k.

Step 3: P(k + 1) = 2^{k+1}

We know that if one number (*i.e.,* element) is added to the elements of a given set, the number of subsets become double.

$\therefore$ Number of subsets of set having (k + 1) distinct elements = $2 \times 2^k = 2^{k+1}$ which is true for P(k + 1). Hence P(k + 1) is true whenever P(k) is true.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer out of the given four options in each of the Exercises from 26 to 28 (M.C.Q.)

Q26. If $10^n + 3.4^{n+2} + k$ is divisible by 9 for all $n \in$ N, then the least positive integral value of k is

(*a*) 5 (*b*) 3 (*c*) 7 (*d*) 1

Sol. Let $P(n) = 10^n + 3.4^{n+2} + k$ is divisible by 9, $\forall\ n \in N$

$P(1) = 10^1 + 3.4^{1+2} + k = 10 + 3.64 + k$

$= 10 + 192 + k = 202 + k$ must be divisible by 9.

If $(202 + k)$ is divisible by 9 then k must be equal to 5

$202 + 5 = 207$ which is divisible by 9

$= \dfrac{207}{9} = 23$

So, the least positive integral value of $k = 5$.

Hence, the correct option is (*a*).

Q27. For all $n \in N$, $3.5^{2n+1} + 2^{3n+1}$

(*a*) 19 (*b*) 17 (*c*) 23 (*d*) 25

Sol. Let $P(n)$: $3.5^{2n+1} + 2^{3n+1}$

For $P(1) : 3.5^{2.1+1} + 2^{3.1+1} = 3.5^3 + 2^4 = 3(125) + 16 = 375 + 16$

$= 391 = 23 \times 17$

So it is divisible by 17 and 23 both.

Hence, the correct option is (*b*) and (*c*).

Q28. If $x^n - 1$ is divisible by $x - k$, then the least positive integral value of k is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

Sol. Let $P(n) = x^n - 1$ is divisible by $x - k$.

$P(1) = x - 1$ is divisible by $x - k$.

Since $k = 1$ is the possible least integral value of k.

Hence, the correct option is (*a*).

Fill in the Blanks in the Exercises 29.

Q29. If $P(n) : 2n < n!$, $n \in N$, then $P(n)$ is true for $n \geq$ ....

Sol. Given that $P(n) : 2n < n!$, $\forall\ n \in N$

For $n = 1$ $\quad 2 < 1$ (Not true)

For $n = 2$ $\quad 2 \times 2 < 2! \Rightarrow 4 < 2$ (Not true)

For $n = 3$ $\quad 2 \times 3 < 3! \Rightarrow 6 < 3.2.1 \Rightarrow 6 < 6$ (Not true)

For $n = 4$ $\quad 2 \times 4 < 4! \Rightarrow 8 < 4.3.2.1 \Rightarrow 8 < 24$ (True)

For $n = 5$ $\quad 2 \times 5 < 5! \Rightarrow 10 < 5.4.3.2.1 \Rightarrow 10 < 120$ (True)

So, $P(n)$ is the true for $n \geq 4$.

Hence, the value of the filler is **4**.

State True or False for the Statements in the Exercises 30.

Q30. Let $P(n)$ be a statement and let $P(k) \Rightarrow P(k + 1)$, for some natural number k, then $P(n)$ is true for all $n \in N$.

Sol. Given that: $\quad P(k) \Rightarrow P(k + 1)$

$P(1) \Rightarrow P(2)$ which is not true.

Hence, the statement is 'False'.