EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. For a positive integer n, find the value of $(1-i)^n\left(1-\frac{1}{i}\right)^n$.

Sol. We have $(1-i)^n\left(1-\frac{1}{i}\right)^n$

$$= \left[(1-i)\left(1-\frac{1}{i}\right)\right]^n = \left[(1-i)\left(1-\frac{1}{i}\times\frac{i}{i}\right)\right]^n = \left[(1-i)\left(1-\frac{i}{i^2}\right)\right]^n$$

$= [(1-i)(1+i)]^n$ $\quad [\because i^2 = -1]$

$= [1-i^2]^n = [1+1]^n = 2^n$

Hence, $(1-i)^n\left(1-\frac{1}{i}\right)^n = 2^n$.

Q2. Evaluate: $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $n \in$ N.

Sol. We have $\sum_{n=1}^{13}(i^n + i^{n+1})$

$= (i+i^2) + (i^2+i^3) + (i^3+i^4) + (i^4+i^5) + (i^5+i^6) + (i^6+i^7) + (i^7+i^8)$
$\quad + (i^8+i^9) + (i^9+i^{10}) + (i^{10}+i^{11}) + (i^{11}+i^{12}) + (i^{12}+i^{13}) + (i^{13}+i^{14})$

$= i + 2(i^2+i^3+i^4+i^5+i^6+i^7+i^8+i^9+i^{10}+i^{11}+i^{12}+i^{13}) + i^{14}$

$= i + 2[-1-i+1+i-1-i+1+i-1-i+1+i] + (-1)$

$= i + 2(0) - 1 \Rightarrow -1+i$

Hence, $\sum_{n=1}^{13}(i^n + i^{n+1}) = -1+i$.

Q3. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$, then find (x, y)

Sol. We have $\quad \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$

$$\Rightarrow \left[\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right]^3 - \left[\frac{(1-i)(1-i)}{(1+i)(1-i)}\right]^3 = x+iy$$

$$\Rightarrow \quad \left[\frac{1+i^2+2i}{1-i^2}\right]^3 - \left[\frac{1+i^2-2i}{1-i^2}\right]^3 = x+iy$$

$$\Rightarrow \quad \left[\frac{1-1+2i}{1+1}\right]^3 - \left[\frac{1-1-2i}{1+1}\right]^3 = x+iy$$

$$\Rightarrow \quad \left(\frac{2i}{2}\right)^3 - \left(\frac{-2i}{2}\right)^3 = x+iy$$

$\Rightarrow \quad (i)^3 - (-i)^3 = x + iy \quad \Rightarrow \quad i^2.i + i^2.i = x + iy$

$\Rightarrow \quad -i - i = x + iy \quad \Rightarrow \quad 0 - 2i = x + iy$

Comparing the real and imaginary parts, we get

$x = 0,\ y = -2$. Hence, $(x, y) = (0, -2)$.

Q4. If $\dfrac{(1+i)^2}{2-i} = x + iy$ then find the value of $x + y$.

Sol. Given that: $\dfrac{(1+i)^2}{2-i} = x + iy \Rightarrow \dfrac{1+i^2+2i}{2-i} = x+iy$

$$\Rightarrow \quad \frac{1-1+2i}{2-i} = x+iy \Rightarrow \frac{2i}{2-i} = x+iy$$

$$\Rightarrow \quad \frac{2i(2+i)}{(2-i)(2+i)} = x+iy \Rightarrow \frac{4i+2i^2}{4-i^2} = x+iy$$

$$\Rightarrow \quad \frac{4i-2}{4+1} = x+iy \qquad [\because i^2 = -1]$$

$$\Rightarrow \quad \frac{-2+4i}{5} = x+iy \Rightarrow \frac{-2}{5} + \frac{4}{5}i = x+iy$$

Comparing the real and imaginary parts, we get

$$x = \frac{-2}{5} \text{ and } y = \frac{4}{5}$$

Hence, $\quad x + y = \dfrac{-2}{5} + \dfrac{4}{5} = \dfrac{2}{5}.$

Q5. If $\left(\dfrac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

Sol. We have $\left(\dfrac{1-i}{1+i}\right)^{100} = a + bi$

$$\Rightarrow \quad \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^{100} = a+bi \Rightarrow \left(\frac{1+i^2-2i}{1-i^2}\right)^{100} = a+bi$$

$$\Rightarrow \quad \left(\frac{1-1-2i}{1+1}\right)^{100} = a+bi \Rightarrow \left(\frac{-2i}{2}\right)^{100} = a+bi$$

$\Rightarrow \quad (-i)^{100} = a + bi \Rightarrow i^{100} = a + bi$

$\Rightarrow \quad (i^4)^{25} = a + bi \Rightarrow (1)^{25} = a + bi \Rightarrow 1 = a + bi$

$\Rightarrow \quad 1 + 0i = a + bi$

Comparing the real and imaginary parts, we have

$a = 1, b = 0$

Hence $(a, b) = (1, 0)$

Q6. If $a = \cos\theta + i\sin\theta$, find the value of $\dfrac{1+a}{1-a}$.

Sol. Given that: $a = \cos\theta + i\sin\theta$

$$\therefore \quad \frac{1+a}{1-a} = \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$

$$= \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} \times \frac{1-\cos\theta + i\sin\theta}{1-\cos\theta + i\sin\theta}$$

$$= \frac{1-\cos\theta + i\sin\theta + \cos\theta - \cos^2\theta + i\sin\theta\cos\theta + i\sin\theta - i\sin\theta\cos\theta + i^2\sin^2\theta}{(1-\cos\theta)^2 - i^2\sin^2\theta}$$

$$= \frac{1 + i\sin\theta - \cos^2\theta + i\sin\theta - \sin^2\theta}{1+\cos^2\theta - 2\cos\theta + \sin^2\theta}$$

$$= \frac{\sin^2\theta + 2i\sin\theta - \sin^2\theta}{1+1-2\cos\theta} = \frac{2i\sin\theta}{2-2\cos\theta}$$

$$= \frac{2i\sin\theta}{2(1-\cos\theta)} = \frac{i\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.i}{2\sin^2\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2}.i$$

Hence, $\dfrac{1+a}{1-a} = i\cot\dfrac{\theta}{2}$.

Q7. If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.

Sol. Given that: $(1+i)z = (1-i)\bar{z}$

$$\Rightarrow \quad \frac{z}{\bar{z}} = \frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{1+i^2-2i}{1-i^2}$$

$$= \frac{1-1-2i}{1+1} = \frac{-2i}{2} = -i$$

$$\Rightarrow \quad \frac{z}{\bar{z}} = -i$$

$\therefore \quad z = -i\bar{z}$. Hence proved.

Q8. If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + b = 0$, where $b \in R$ represents a circle.

Sol. Given that: $z = x + iy$

To prove: $z\bar{z} + 2(z + \bar{z}) + b = 0$

$\Rightarrow (x + iy)(x - iy) + 2(x + iy + x - iy) + b = 0$

$\Rightarrow x^2 + y^2 - 2(x + x) + b = 0$

$\Rightarrow x^2 + y^2 - 4x + b = 0$

Which represents a circle. Hence proved.

Q9. If the real part of $\dfrac{\bar{z} + 2}{\bar{z} - 1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

Sol. Let $z = x + iy$

$\therefore \bar{z} = x - iy$

So
$$\frac{\bar{z} + 2}{\bar{z} - 1} = \frac{x - iy + 2}{x - iy - 1}$$

$$= \frac{(x + 2) - iy}{(x - 1) - iy} = \frac{(x + 2) - iy}{(x - 1) - iy} \times \frac{(x - 1) + iy}{(x - 1) + iy}$$

$$= \frac{(x + 2)(x - 1) + (x + 2)yi - (x - 1)yi - i^2y^2}{(x - 1)^2 - i^2y^2}$$

$$= \frac{x^2 + 2x - x - 2 + (x + 2 - x + 1)yi + y^2}{(x - 1)^2 + y^2}$$

$$= \frac{x^2 + y^2 + x - 2}{(x - 1)^2 + y^2} + \frac{3y}{(x - 1)^2 + y^2}i$$

Real part = 4

$$\therefore \frac{x^2 + y^2 + x - 2}{(x - 1)^2 + y^2} = 4$$

$\Rightarrow x^2 + y^2 + x - 2 = 4[(x - 1)^2 + y^2]$

$\Rightarrow x^2 + y^2 + x - 2 = 4[x^2 + 1 - 2x + y^2]$

$\Rightarrow x^2 + y^2 + x - 2 = 4x^2 + 4 - 8x + 4y^2$

$\Rightarrow x^2 - 4x^2 + y^2 - 4y^2 + x + 8x - 2 - 4 = 0$

$\Rightarrow -3x^2 - 3y^2 + 9x - 6 = 0$

$\Rightarrow x^2 + y^2 - 3x + 2 = 0$

Which represents a circle. Hence, z lies on a circle.

Q10. Show that the complex number z, satisfying the condition $\arg\left(\dfrac{z - 1}{z + 1}\right) = \dfrac{\pi}{4}$ lies on a circle.

Sol. Let $z = x + iy$

Given that: $\arg\left(\dfrac{z - 1}{z + 1}\right) = \dfrac{\pi}{4}$

$$\Rightarrow \quad \arg(z-1) - \arg(z+1) = \frac{\pi}{4}$$

$$\left[\because \arg(z_1) - \arg(z_2) = \arg\frac{z_1}{z_2}\right]$$

$$\Rightarrow \quad \arg[x + iy - 1] - \arg[x + iy + 1] = \frac{\pi}{4}$$

$$\Rightarrow \quad \arg[(x-1) + iy] - \arg[(x+1) + iy] = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x-1} - \tan^{-1}\frac{y}{x+1} = \frac{\pi}{4}$$

$$\left[\because \arg(x + yi) = \tan^{-1}\frac{y}{x}\right]$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y}{x-1} \times \frac{y}{x+1}}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{xy + y - xy + y}{x^2 - 1 + y^2} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2y}{x^2 + y^2 - 1} = 1$$

$$\Rightarrow \quad x^2 + y^2 - 1 = 2y$$

$\Rightarrow \quad x^2 + y^2 - 2y - 1 = 0$ which is a circle.

Hence, z lies on a circle.

Q11. Solve the equation $|z| = z + 1 + 2i$

Sol. Given that: $|z| = z + 1 + 2i$

Let $z = x + iy$

$$|z| = (z + 1) + 2i$$

Squaring both sides

$$|z|^2 = |z+1|^2 + 4i^2 + 4(z+1)i$$

$$\Rightarrow \quad |z|^2 = |z|^2 + 1 + 2z - 4 + 4(z+1)i$$

$$\Rightarrow \quad 0 = -3 + 2z + 4(z+1)i$$

$$\Rightarrow \quad 3 - 2z - 4(z+1)i = 0$$

$$\Rightarrow \quad 3 - 2(x + yi) - 4[x + yi + 1]i = 0$$

$$\Rightarrow \quad 3 - 2x - 2yi - 4xi - 4yi^2 - 4i = 0$$

$$\Rightarrow \quad 3 - 2x + 4y - 2yi - 4i - 4xi = 0$$

$$\Rightarrow \quad (3 - 2x + 4y) - i(2y + 4x + 4) = 0$$

$\Rightarrow \quad 3 - 2x + 4y = 0 \Rightarrow 2x - 4y = 3$ ...(*i*)

and $\quad 4x + 2y + 4 = 0 \Rightarrow 2x + y = -2$ ...(*ii*)

Solving eqn. (*i*) and (*ii*), we get

$y = -1$ and $x = -\dfrac{1}{2}$

Hence, the value of $z = x + yi = \left(-\dfrac{1}{2} - i\right)$.

LONG ANSWER TYPE QUESTIONS

Q12. If $|z+1| = z + 2(1 + i)$ then find z.

Sol. Given that: $|z+1| = z + 2(1 + i)$

Let $z = x + iy$

So, $|x + iy + 1| = (x + iy) + 2(1 + i)$

$\Rightarrow \quad |(x+1) + iy| = x + iy + 2 + 2i$

$\Rightarrow \quad |(x+1) + iy| = (x + 2) + (y + 2)i$

$\Rightarrow \quad \sqrt{(x+1)^2 + y^2} = (x + 2) + (y + 2)i \qquad \left[\because |x+iy| = \sqrt{x^2+y^2}\right]$

Squaring both sides, we get

$$(x + 1)^2 + y^2 = (x + 2)^2 + (y + 2)^2.i^2 + 2(x + 2)(y + 2)i$$

$$\Rightarrow x^2 + 1 + 2x + y^2 = x^2 + 4 + 4x - y^2 - 4y - 4 + 2(x + 2)(y + 2)i$$

Comparing the real and imaginary parts, we get

$$x^2 + 1 + 2x + y^2 = x^2 + 4x - y^2 - 4y \text{ and } 2(x + 2)(y + 2) = 0$$

$\Rightarrow \quad 2y^2 - 2x + 4y + 1 = 0 \qquad$ ...(*i*)

and $\quad (x + 2)(y + 2) = 0 \qquad$ ...(*ii*)

$x + 2 = 0$ or $y + 2 = 0$

$\therefore \quad x = -2$ or $y = -2$

Now put $x = -2$ in eqn. (*i*)

$2y^2 - 2 \times (-2) + 4y + 1 = 0$

$\Rightarrow \quad 2y^2 + 4 + 4y + 1 = 0$

$\Rightarrow \quad 2y^2 + 4y + 5 = 0$

$b^2 - 4ac = (4)^2 - 4 \times 2 \times 5$

$= 16 - 40 = -24 < 0$ no real roots.

Put $y = -2$ in eqn. (*i*)

$2(-2)^2 - 2x + 4(-2) + 1 = 0$

$$8 - 2x - 8 + 1 = 0 \Rightarrow x = \frac{1}{2} \text{ and } y = -2$$

Hence, $z = x + iy = \left(\dfrac{1}{2} - 2i\right)$.

Q13. If $\arg(z - 1) = \arg(z + 3i)$ then find $x - 1 : y$ where $z = x + iy$.

Sol. Given that: $\arg(z - 1) = \arg(z + 3i)$

$\Rightarrow \quad \arg[x + yi - 1] = \arg[x + yi + 3i]$

$\Rightarrow \quad \arg[(x - 1) + yi] = \arg[x + (y + 3)i]$

$$\Rightarrow \quad \tan^{-1}\frac{y}{x-1} = \tan^{-1}\frac{y+3}{x}$$

$$\Rightarrow \quad \frac{y}{x-1} = \frac{y+3}{x}$$

$$\Rightarrow \quad xy = (x-1)(y+3) \Rightarrow xy = xy + 3x - y - 3$$

$$\Rightarrow \quad 3x - y = 3 \Rightarrow 3x - 3 = y$$

$$\Rightarrow \quad 3(x-1) = y \Rightarrow \frac{(x-1)}{y} = \frac{1}{3} \Rightarrow x - 1 : y = 1 : 3$$

Hence, $x - 1 : y = 1 : 3$.

Q14. Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.

Sol. Given that: $\left|\frac{z-2}{z-3}\right| = 2$

Let $z = x + iy$

$$\therefore \quad \left|\frac{x+iy-2}{x+iy-3}\right| = 2 \Rightarrow \left|\frac{(x-2)+iy}{(x-3)+iy}\right| = 2$$

$$\Rightarrow \quad \frac{|(x-2)+iy|}{|(x-3)+iy|} = 2 \qquad \left[\because \left|\frac{a}{b}\right| = \frac{|a|}{|b|}\right]$$

$$\Rightarrow \quad |(x-2)+iy| = 2|(x-3)+iy|$$

$$\Rightarrow \quad \sqrt{(x-2)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$$

Squaring both sides, we get

$$(x-2)^2 + y^2 = 4[(x-3)^2 + y^2]$$

$$\Rightarrow \quad x^2 + 4 - 4x + y^2 = 4[x^2 + 9 - 6x + y^2]$$

$$\Rightarrow \quad x^2 + y^2 - 4x + 4 = 4x^2 + 4y^2 - 24x + 36$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

Here $g = \frac{-10}{3}$, $f = 0$,

$$r = \sqrt{g^2 + f^2 - c} = \sqrt{\frac{100}{9} + 0 - \frac{32}{3}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

Hence, the required equation of the circle is

$$x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

Centre $= (-g, -f) = \left(\frac{10}{3}, 0\right)$ and $r = \frac{2}{3}$.

Q15. If $\frac{z-1}{z+1}$ is purely imaginary number ($z \neq -1$), then find the value of $|z|$.

Sol. Given that $\frac{z-1}{z+1}$ is purely imaginary number

Let $z = x + yi$

$$\therefore \quad \frac{x+yi-1}{x+yi+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$\Rightarrow \frac{(x-1)(x+1)-iy(x-1)+(x+1)iy-i^2y^2}{(x+1)^2-i^2y^2}$$

$$\Rightarrow \frac{x^2-1+iy(x+1-x+1)+y^2}{x^2+1+2x+y^2} = \frac{x^2+y^2-1+2yi}{x^2+y^2+2x+1}$$

$$\Rightarrow \frac{x^2+y^2-1}{x^2+y^2+2x+1} + \frac{2y}{x^2+y^2+2x+1}i$$

Since, the number is purely imaginary, then real part = 0

$$\therefore \quad \frac{x^2+y^2-1}{x^2+y^2+2x+1} = 0$$

$$\Rightarrow \quad x^2+y^2-1 = 0 \quad \Rightarrow x^2+y^2 = 1$$

$$\Rightarrow \quad \sqrt{x^2+y^2} = 1 \quad \therefore |z| = 1$$

Q16. z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\bar{z}_2$.

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are polar form of two complex numbers z_1 and z_2.

Given that: $|z_1| = |z_2| \Rightarrow r_1 = r_2$ ...(*i*)

and $\arg(z_1) + \arg(z_2) = \pi$

$$\Rightarrow \quad \theta_1 + \theta_2 = \pi$$

$$\Rightarrow \quad \theta_1 = \pi - \theta_2$$

Now $z_1 = r_1[\cos(\pi-\theta_2) + i\sin(\pi-\theta_2)]$

$$\Rightarrow \quad z_1 = r_1[-\cos\theta_2 + i\sin\theta_2]$$

$$\Rightarrow \quad z_1 = -r_1(\cos\theta_2 - i\sin\theta_2) \quad ...(i)$$

$$z_2 = r_2[\cos\theta_2 + i\sin\theta_2]$$

$$\bar{z}_2 = r_1[\cos\theta_2 - i\sin\theta_2] \quad [\because r_1 = r_2] \quad ...(ii)$$

From eqn. (*i*) and (*ii*) we get,

$z_1 = -\bar{z}_2$. Hence proved.

Q17. If $|z_1| = 1$ $(z_1 \neq -1)$ and $z_2 = \dfrac{z_1 - 1}{z_1 + 1}$, then show that the real part of z_2 is 0.

Sol. Let $z_1 = x + yi$

$$|z_1| = \sqrt{x^2 + y^2} = 1 \qquad \text{[given that } |z_1| = 1\text{]}$$

$$\Rightarrow \quad x^2 + y^2 = 1 \qquad \ldots(i)$$

Now
$$z_2 = \frac{z_1 - 1}{z_1 + 1} = \frac{x + yi - 1}{x + yi + 1}$$

$$= \frac{(x-1) + yi}{(x+1) + yi} = \frac{(x-1) + yi}{(x+1) + yi} \times \frac{x + 1 - yi}{x + 1 - yi}$$

$$= \frac{(x-1)(x+1) - y(x-1)i + y(x+1)i - y^2 i^2}{(x+1)^2 - y^2 i^2}$$

$$= \frac{x^2 - 1 + yi\,(x + 1 - x + 1) + y^2}{x^2 + 1 + 2x + y^2}$$

$$= \frac{(x^2 + y^2 - 1) + 2yi}{x^2 + y^2 + 2x + 1}$$

$$= \frac{(1 - 1)}{x^2 + y^2 + 2x + 1} + \frac{2y}{x^2 + y^2 + 2x + 1} i$$

$$= 0 + \frac{2y}{x^2 + y^2 + 2x + 1} i$$

Hence, the real part of z_2 is 0.

Q18. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then find $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right)$.

Sol. Let the polar form of $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

$\therefore \quad z_2 = \overline{z}_1 = r_1 (\cos \theta_1 - i \sin \theta_1) = r_1 [\cos(-\theta_1) + i \sin(-\theta_1)]$

Similarly, $\quad z_3 = r_2 (\cos \theta_2 + i \sin \theta_2)$

$\therefore \quad z_4 = \overline{z}_3 = r_2 (\cos \theta_2 - i \sin \theta_2) = r_2 [\cos(-\theta_2) + i \sin(-\theta_2)]$

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3)$$
$$= \theta_1 - (-\theta_2) + (-\theta_1) - \theta_2$$
$$= \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0$$

Hence, $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right) = 0$.

Q19. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, then show that

$$|z_1 + z_2 + z_3 + \ldots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \ldots + \frac{1}{z_n}\right|.$$

Sol. We have $|z_1| = |z_2| = ... = |z_n| = 1$

$\Rightarrow$ $|z_1|^2 = |z_2|^2 = ... = |z_n|^2 = 1$ ...(*i*)

$\Rightarrow$ $z_1\bar{z}_1 = z_2\bar{z}_2 = ... = z_n\bar{z}_n = 1$ $[\because z\bar{z} = |z|^2]$

$\Rightarrow$ $z_1 = \dfrac{1}{\bar{z}_1},\ z_2 = \dfrac{1}{\bar{z}_2} = ... = z_n = \dfrac{1}{\bar{z}_n}$

L.H.S. $|z_1 + z_2 + z_3 + ... + z_n|$

$$= \left|\frac{z_1\bar{z}_1}{\bar{z}_1} + \frac{z_2\bar{z}_2}{\bar{z}_2} + \frac{z_3\bar{z}_3}{\bar{z}_3} + ... + \frac{z_n\bar{z}_n}{\bar{z}_n}\right|$$

$$= \left|\frac{|z_1|^2}{\bar{z}_1} + \frac{|z_2|^2}{\bar{z}_2} + \frac{|z_3|^2}{\bar{z}_3} + ... + \frac{|z_n|^2}{\bar{z}_n}\right| \qquad [z\bar{z} = |z|^2]$$

$$= \left|\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + ... + \frac{1}{\bar{z}_n}\right| \qquad [\text{using } (i)]$$

$$= \left|\overline{\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + ... + \frac{1}{z_n}}\right| \qquad [\because \bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}]$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + ... + \frac{1}{z_n}\right| \qquad [\because |z| = |\bar{z}|]$$

L.H.S. = R.H.S. Hence proved.

Q20. If for complex numbers z_1 and z_2, arg (z_1) – arg (z_2) = 0, then show that $|z_1 - z_2| = |z_1| - |z_2|$.

Sol. Given that for z_1 and z_2, arg (z_1) – arg (z_2) = 0
Let us represent z_1 and z_2 in polar form
$z_1 = r_1 (\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$
arg $(z_1) = \theta_1$ and arg $(z_2) = \theta_2$
Since arg (z_1) – arg (z_2) = 0
$\Rightarrow \theta_1 - \theta_2 = 0 \Rightarrow \theta_1 = \theta_2$
Now $z_1 - z_2 = r_1(\cos\theta_1 + i\sin\theta_1) - r_2(\cos\theta_2 + i\sin\theta_2)$
$= r_1\cos\theta_1 + i\,r_1\sin\theta_1 - r_2\cos\theta_1 - i\,r_2\sin\theta_1$ $[\because \theta_1 = \theta_2]$
$= (r_1\cos\theta_1 - r_2\cos\theta_1) + i(r_1\sin\theta_1 - r_2\sin\theta_1)$

$$\therefore\ |z_1 - z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_1)^2 + (r_1\sin\theta_1 - r_2\sin\theta_1)^2}$$

$$= \sqrt{(r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_1 - 2r_1r_2\cos^2\theta_1 + r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_1 - 2r_1r_2\sin^2\theta_1)}$$

$$= \sqrt{r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_1 + \sin^2\theta_1) - 2r_1r_2(\cos^2\theta_1 + \sin^2\theta_1)}$$

$= \sqrt{r_1^2 + r_2^2 - 2r_1r_2} = \sqrt{(r_1 - r_2)^2} = r_1 - r_2$

$= |z_1| - |z_2|$

Hence, $|z_1 - z_2| = |z_1| - |z_2|$

Q21. Solve the system of equations $\text{Re}(z^2) = 0$, $|z| = 2$.

Sol. Given that: $\text{Re}(z^2) = 0$ and $|z| = 2$.

Let $z = x + yi$

$\therefore \quad |z| = \sqrt{x^2 + y^2}$

$\Rightarrow \quad \sqrt{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 = 4$ ...(*i*)

Since, $z = x + yi$

$z^2 = x^2 + y^2i^2 + 2xyi \quad \Rightarrow \quad z^2 = x^2 - y^2 + 2xyi$

$\therefore \quad \text{Re}(z^2) = x^2 - y^2$

$\Rightarrow \quad x^2 - y^2 = 0$ ...(*ii*)

From eqn. (*i*) and (*ii*), we get

$x^2 + y^2 = 4 \Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$

Hence, $z = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2}$.

Q22. Find the complex number satisfying the equation

$z + \sqrt{2}\,|(z + 1)| + i = 0$

Sol. Given that: $z + \sqrt{2}\,|(z + 1)| + i = 0$

Let $z = x + yi$

$\therefore \quad (x + yi) + \sqrt{2}\,|(x + yi + 1)| + i = 0$

$\Rightarrow \quad x + (y + 1)i + \sqrt{2}\,|(x + 1) + yi)| = 0$

$\Rightarrow \quad x + (y + 1)i + \sqrt{2}\sqrt{(x + 1)^2 + y^2} = 0$

$\Rightarrow \quad x + (y + 1)i + \sqrt{2}\sqrt{x^2 + 2x + 1 + y^2} = 0 + 0i$

$\Rightarrow \quad x + \sqrt{2}\sqrt{x^2 + 2x + 1 + y^2} = 0, \; y + 1 = 0$

$\Rightarrow \quad x = -\sqrt{2}\sqrt{x^2 + 2x + 1 + y^2}$ and $y = -1$

$\Rightarrow \quad x^2 = 2(x^2 + 2x + 1 + y^2)$

$\Rightarrow \quad x^2 = 2x^2 + 4x + 2 + 2y^2$

$\Rightarrow \quad x^2 + 4x + 2 + 2y^2 = 0$

$\Rightarrow \quad x^2 + 4x + 2 + 2(-1)^2 = 0$ $[\because y = -1]$

$\Rightarrow \quad x^2 + 4x + 4 = 0$

$\Rightarrow \quad (x + 2)^2 = 0$

$\Rightarrow \quad x + 2 = 0 \Rightarrow x = -2$

Hence, $z = x + yi = -2 - i$.

Q23. Write the complex number $z = \dfrac{1-i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$ in polar form.

Sol. Given that:

$$z = \frac{1-i}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \frac{2-2i}{1+i\sqrt{3}} = \frac{2-2i}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$\Rightarrow \quad z = \frac{2-2\sqrt{3}i-2i+2\sqrt{3}i^2}{(1)^2-(i\sqrt{3})^2} = \frac{2-2\sqrt{3}i-2i-2\sqrt{3}}{1-3i^2}$$

$$= \frac{(2-2\sqrt{3})-(2+2\sqrt{3})i}{4} = \frac{1-\sqrt{3}}{2} - \frac{1+\sqrt{3}}{2}i$$

$$r = \sqrt{\left(\frac{1-\sqrt{3}}{2}\right)^2 + \left(-\frac{1+\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1+3-2\sqrt{3}}{4} + \frac{1+3+2\sqrt{3}}{4}}$$

$$= \sqrt{\frac{4-2\sqrt{3}+4+2\sqrt{3}}{4}} = \sqrt{\frac{8}{4}} = \sqrt{2}$$

So $r = \sqrt{2}$

Now $\arg(z) = \tan^{-1}\dfrac{y}{x}$

$$\Rightarrow \theta = \tan^{-1}\frac{-\left(\frac{1+\sqrt{3}}{2}\right)}{\left(\frac{1-\sqrt{3}}{2}\right)} = \tan^{-1}\left[-\left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)\right] = \tan^{-1}\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$\Rightarrow \theta = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right] \quad \left[\because \tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}}\right]$$

$$\Rightarrow \quad \theta = \frac{5\pi}{12}$$

Hence, the polar is

$$z = \sqrt{2}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right].$$

Q24. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \dfrac{\pi}{2}$, then show that $\bar{z}w = -i$.

Sol. Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$

$$zw = r_1r_2[(\cos\theta_1 + i\sin\theta_1)][(\cos\theta_2 + i\sin\theta_2)]$$

$|zw| = r_1 r_2 = 1$ (given)

Now $\arg(z) - \arg(w) = \dfrac{\pi}{2}$

$$\theta_1 - \theta_2 = \frac{\pi}{2} \Rightarrow \arg\left(\frac{z}{w}\right) = \frac{\pi}{2}$$

$$\bar{z}w = r_1(\cos\theta_1 - i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 - i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2]$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)]$$

$$= r_1 r_2[\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)]$$

$$= r_1 r_2\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$$

$$= r_1 r_2\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right] = 1.[0 - i]$$

Here $\bar{z}w = -i$. Hence proved.

Fill in the Blanks in Each of the Exercises 25.

Q25.

(*i*) For any two complex numbers z_1, z_2 and any real numbers a, b, $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$ ....

(*ii*) The value of $\sqrt{-25} \times \sqrt{-9}$ is ....

(*iii*) The number $\dfrac{(1-i)^3}{1-i^3}$ is equal to ....

(*iv*) The sum of series $i + i^2 + i^3 + \ldots$ upto 1000 terms is ....

(*v*) Multiplicative inverse of $1 + i$ is ....

(*vi*) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_2 =$ ....

(*vii*) $\arg(z) + \arg(\bar{z})$ $(\bar{z} \neq 0)$ is ....

(*viii*) If $|z+4| \leq 3$, then the greatest and least values of $|z+1|$ are .... and ....

(*ix*) If $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$, then the locus of z is ....

(*x*) If $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$, then $z =$ ....

Sol. (*i*) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$$= |az_1|^2 + |bz_2|^2 - 2\,\mathrm{Re}(az_1 . b\bar{z}_2) + |bz_1|^2 + |az_2|^2 + 2\,\mathrm{Re}(az_1 . b\bar{z}_2)$$

$$= |az_1|^2 + |bz_2|^2 + |bz_1|^2 + |az_2|^2$$

$= (a^2 + b^2)\left(|z_1|^2 + |z_2|^2\right)$

Hence, the value of the filler is $\boldsymbol{(a^2 + b^2)\left(|z_1|^2 + |z_2|^2\right)}$.

(*ii*) $\sqrt{-25} \times \sqrt{-9} = \sqrt{-1}.\sqrt{25} \times \sqrt{-1}.\sqrt{9}$
$= 5i \times 3i = 15i^2 = -15$

Hence, the value of the filler is **– 15**.

(*iii*) $$\frac{(1-i)^3}{1-i^3} = \frac{(1-i)^3}{(1-i)(1+i+i^2)} = \frac{(1-i)^2}{(1+i-1)} = \frac{1+i^2-2i}{i}$$
$$= \frac{1-1-2i}{i} = \frac{-2i}{i} = -2$$

Hence, the value of the filler is **– 2**.

(*iv*) $i + i^2 + i^3 + \ldots$ upto 1000 terms
$= i + i^2 + i^3 + \ldots + i^{1000} = 0$

$$\left[\sum_{n=1}^{1000} i^n = 0\right]$$

Hence, the value of the filler is **0**.

(*v*) Multiplicative inverse of

$$1 + i = \frac{1}{1+i} = \frac{1 \times (1-i)}{(1+i)(1-i)}$$
$$= \frac{1-i}{1-i^2} = \frac{1-i}{1+1} = \frac{1}{2}(1-i)$$

Hence, the value of the filler = $\boldsymbol{\frac{1}{2}(1-i)}$.

(*vi*) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$

If $z_1 + z_2$ is real then

$y_1 + y_2 = 0 \Rightarrow y_1 = -y_2$

$\therefore$ $z_2 = x_2 - iy_1$

$z_2 = x_1 - iy_1$ (when $x_1 = x_2$)

So $z_2 = \overline{z}_1$

Hence, the value of the filler is $\boldsymbol{\overline{z}_1}$.

(*vii*) $\arg(z) + \arg(\overline{z})$ $(\overline{z} \neq 0)$

If $\arg(z) = \theta$, then $\arg(\overline{z}) = -\theta$

So $\theta + (-\theta) = 0$

Hence, the value of the filler is **0**.

(*viii*) Given that: $|z + 4| \leq 3$

For the greatest value of

$$|z+1| = |z+4-3| \le |z+4| + |-3|$$
$$= |z+4-3| \le 3+3 \quad [\because |z+4| \le 3 \text{ and } |-3| = 3]$$
$$= |z+4-3| \le 6$$

Hence, the greatest value of $|z+1|$ is 6 and for the least value of $|z+1| = 0$.

[$\because$ The least value of the modulus of complex number is 0]

Hence, the value of the filler are **6** and **0**.

(*ix*) Given that: $\dfrac{z-2}{z+2} = \dfrac{\pi}{6}$

Let $z = x + iy$

$$\Rightarrow \quad \left|\frac{x+iy-2}{x+iy+2}\right| = \frac{\pi}{6} \Rightarrow \left|\frac{(x-2)+iy}{(x+2)+iy}\right| = \frac{\pi}{6}$$
$$\Rightarrow \quad 6|(x-2)+iy| = \pi|(x+2)+iy|$$
$$\Rightarrow \quad 6\sqrt{(x-2)^2+y^2} = \pi\sqrt{(x+2)^2+y^2}$$
$$\Rightarrow \quad 36[x^2+4-4x+y^2] = \pi^2[x^2+4+4x+y^2]$$
$$\Rightarrow 36x^2+144-144x+36y^2 = \pi^2x^2+4\pi^2+4\pi^2x+\pi^2y^2$$
$$\Rightarrow (36-\pi^2)x^2+(36-\pi^2)y^2-(144+4\pi^2)x+144-4\pi^2 = 0$$

Which represents are equation of a circle.

Hence, the value of the filler is **circle**.

(*x*) Given that: $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$

Let $z = x + yi$

$$|z| = \sqrt{x^2+y^2} = 4$$
$$\Rightarrow \quad x^2+y^2 = 16 \qquad ...(i)$$
$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \frac{5\pi}{6}$$
$$\Rightarrow \quad \frac{y}{x} = \tan\frac{5\pi}{6} = \tan\left(\pi-\frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$$
$$\therefore \quad x = -\sqrt{3}y \qquad ...(ii)$$

From eqn. (*i*) and (*ii*)

$$\left(-\sqrt{3}y\right)^2+y^2 = 16 \Rightarrow 3y^2+y^2 = 16 \Rightarrow 4y^2 = 16$$
$$\Rightarrow \quad y^2 = 4 \Rightarrow y = \pm 2 \quad \therefore x = -2\sqrt{3}$$

So, $\quad z = -2\sqrt{3}+2i$

Hence, the value of the filler is $\mathbf{-2\sqrt{3}+2i}$

State True or False for the Statements in Each of the Exercises 26.

Q26.

(*i*) The order relation is defined on the set of complex numbers.

(*ii*) Multiplication of a non-zero complex number by $-i$ rotates the point about origin through a right angle in the anti-clockwise direction.

(*iii*) For any complex number z, the minimum value of $|z|+|z-1|$ is 1.

(*iv*) The locus represented by $|z-1|=|z-i|$ is a line perpendicular to the join of the points (1, 0) and (0, 1).

(*v*) If z is a complex number such that $z \neq 0$ and $\text{Re}(z) = 0$ then $\text{lm}\,(z^2) = 0$.

(*vi*) The inequality $|z-4|<|z-2|$ represents the region given by $x > 3$.

(*vii*) Let z_1 and z_2 be two complex numbers such that $|z_1+z_2| = |z_1|+|z_2|$, then $\arg(z_1) - \arg(z_2) = 0$.

(*viii*) 2 is not a complex number.

Sol. (*i*) Comparison of two purely imaginary complex numbers is not possible. However, the two purely real complex numbers can be compared.

So it is 'False'.

(*ii*) Let $z = x + yi$

$z.i = (x + yi)\, i = xi - y$ which rotates at angle of 180°

So, it is 'False'.

(*iii*) Let $z = x + yi$

$\therefore\ |z|+|z-1| = \sqrt{x^2+y^2} + \sqrt{(x-1)^2+y^2}$

The value of $|z|+|z-1|$ is minimum when $x = 0$, $y = 0$ i.e., 1.

Hence, it is 'True'.

(*iv*) Let $z = x + yi$

Given that: $|z-1| = |z-i|$

then $|x+yi-1| = |x+yi-i|$

$\Rightarrow\ |(x-1)+yi| = |x-(1-y)i|$

$\Rightarrow\ \sqrt{(x-1)^2+y^2} = \sqrt{x^2+(1-y)^2}$

$\Rightarrow\ (x-1)^2 + y^2 = x^2 + (1-y)^2$

$\Rightarrow\ x^2 - 2x + 1 + y^2 = x^2 + 1 + y^2 - 2y$

$\Rightarrow\ -2x + 2y = 0$

$\Rightarrow\ x - y = 0$ which is a straight line.

Slope = 1

Now equation of a line through the point (1, 0) and (0, 1)

$$y - 0 = \frac{1-0}{0-1}(x-1)$$

$\Rightarrow\ y = -x + 1$ whose slope $= -1$.

Now the multiplication of the slopes of two lines $= -1 \times 1 = -1$, so they are perpendicular.

Hence, it is 'True'.

(*v*) Let $z = x + yi$, $z \neq 0$ and $\text{Re}(z) = 0$

Since real part is 0 $\Rightarrow x = 0$

$\therefore z = 0 + yi = yi$

$\therefore \text{Im}(z^2) = y^2 i^2 = -y^2$ which is real.

Hence, it is 'False'.

(*vi*) Given that: $|z - 4| < |z - 2|$

Let $z = x + yi$

$$\Rightarrow |x + yi - 4| < |x + yi - 2| \Rightarrow |(x-4) + yi| < |(x-2) + yi|$$

$$\Rightarrow \sqrt{(x-4)^2 + y^2} < \sqrt{(x-2)^2 + y^2}$$

$$\Rightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2 \Rightarrow (x-4)^2 < (x-2)^2$$

$$\Rightarrow x^2 + 16 - 8x < x^2 + 4 - 4x \Rightarrow -8x + 4x < -16 + 4$$

$$\Rightarrow -4x < -12 \Rightarrow x > 3$$

Hence, it is 'True'.

(*vii*) Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

$$\Rightarrow |z_1 + z_2| = |z_1| + |z_2|$$

$$\Rightarrow |x_1 + y_1 i + x_2 + y_2 i| = |x_1 + y_1 i| + |x_2 + y_2 i|$$

$$\Rightarrow |(x_1 + x_2) + (y_1 + y_2)i| = |(x_1 + y_1 i)| + |(x_2 + y_2 i)|$$

$$\Rightarrow \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

Squaring both sides, we get

$$\Rightarrow (x_1 + x_2)^2 + (y_1 + y_2)^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

$$\Rightarrow x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2$$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2}$$

$$\Rightarrow 2x_1x_2 + 2y_1y_2 = 2\sqrt{x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2}$$

$$\Rightarrow x_1x_2 + y_1y_2 = \sqrt{x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2}$$

Again squares on both sides, we get

$$x_1^2x_2^2 + y_1^2y_2^2 + 2x_1y_1x_2y_2 = x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2$$

$$\Rightarrow 2x_1y_1x_2y_2 = x_1^2y_2^2 + x_2^2y_1^2$$

$$\Rightarrow x_1^2y_2^2 + x_2^2y_1^2 - 2x_1y_1x_2y_2 = 0$$

$$\Rightarrow (x_1y_2 - x_2y_1)^2 = 0 \Rightarrow x_1y_2 - x_2y_1 = 0$$

$$\Rightarrow x_1y_2 = x_2y_1 \Rightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2} \Rightarrow \frac{y_1}{x_1} = \frac{y_2}{x_2}$$

$$\Rightarrow \arg(z_1) = \arg(z_2)$$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = 0$

Hence, it is 'True'.

(*viii*) Since 2 has no imaginary part.

So, 2 is not a complex number.

Hence, it is 'True'.

Q27. Match the statements of Column A and Column B.

	Column A		**Column B**
(*a*)	The polar form of $i+\sqrt{3}$ is	(*i*)	Perpendicular bisector of segment joining (– 2, 0) and (2, 0)
(*b*)	The amplitude of $-1+\sqrt{-3}$ is	(*ii*)	On or outside the circle having centre at (0, – 4) and radius 3.
(*c*)	If $\|z+2\|=\|z-2\|$, then real of z is	(*iii*)	$\frac{2\pi}{3}$
(*d*)	If $\|z+2i\|=\|z-2i\|$, then locus of z is	(*iv*)	Perpendicular bisector of segment joining (0, – 2) and (0, 2)
(*e*)	Region represented by $\|z+4i\| \geq 3$ is	(*v*)	$2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$
(*f*)	Region represented by $\|z+4\| \leq 3$	(*vi*)	On or inside the circle having centre (– 4, 0) and radius 3 units
(*g*)	Conjugate of $\frac{1+2i}{1-i}$ lies in	(*vii*)	First quadrant
(*h*)	Reciprocal of $1 - i$ lies in	(*viii*)	Third quadrant

Sol. (*a*) Given that $z = i + \sqrt{3}$

Polar form of $z = r[\cos\theta + i\sin\theta]$

$\Rightarrow \quad \sqrt{3} + i = r\cos\theta + ri\sin\theta$

$\Rightarrow \quad r = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$

and $\quad \tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$

Since $x > 0, y > 0$

$\therefore$ Polar form of $z = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

Hence, $(a) \leftrightarrow (v)$.

(*b*) Given that $z = -1 + \sqrt{-3} = -1 + \sqrt{3}i$

Here argument $(z) = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \tan^{-1}\left|\sqrt{3}\right| = \frac{\pi}{3}$

So, $\alpha = \frac{\pi}{3}$

Since $x < 0$ and $y > 0$

Then $\theta = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Hence, $(b) \leftrightarrow (iii)$.

(*c*) Given that: $|z+2| = |z-2|$

Let $z = x + yi$

$\therefore \quad |x+yi+2| = |x+yi-2| \Rightarrow |(x+2)+yi| = |(x-2)+yi|$

$\Rightarrow \sqrt{(x+2)^2+y^2} = \sqrt{(x-2)^2+y^2}$

$\Rightarrow (x+2)^2 + y^2 = (x-2)^2 + y^2 \Rightarrow (x+2)^2 = (x-2)^2$

$\Rightarrow x^2 + 4 + 4x = x^2 + 4 - 4x \Rightarrow 8x = 0 \Rightarrow x = 0$

Which represent equation of y-axis and it is perpendicular to the line joining the points (– 2, 0) and (2, 0).

Hence, $(c) \leftrightarrow (i)$.

(*d*) $|z+2i| = |z-2i|$

Let $z = x + yi$

$\therefore \quad |x+yi+2i| = |x+yi-2i| \Rightarrow |x+(y+2)i| = |x+(y-2)i|$

$\Rightarrow \sqrt{x^2+(y+2)^2} = \sqrt{x^2+(y-2)^2}$

$\Rightarrow x^2 + (y+2)^2 = x^2 + (y-2)^2 \Rightarrow (y+2)^2 = (y-2)^2$

$\Rightarrow y^2 + 4 + 4y = y^2 + 4 - 4y$

$\Rightarrow 8y = 0 \Rightarrow y = 0$. Which is the equation of x-axis and it is perpendicular to the line segment joining (0, – 2) and (0, 2).

Hence, $(d) \leftrightarrow (iv)$.

(*e*) Given that: $|z+4i| \geq 3$

Let $z = x + yi$

$\therefore \quad |x+yi+4i| \geq 3 \Rightarrow |x+(y+4)i| \geq 3$

$\Rightarrow \sqrt{x^2+(y+4)^2} \geq 3 \Rightarrow x^2 + (y+4)^2 \geq 9$

$\Rightarrow x^2 + y^2 + 8y + 16 \geq 9 \Rightarrow x^2 + y^2 + 8y + 7 \geq 0$

$\Rightarrow r = \sqrt{(4)^2 - 7} = 3$

Which represents a circle on or outside having centre (0, – 4) and radius 3.

Hence, (*e*) ↔ (*ii*).

(*f*) $|z+4| \le 3$

Let $z = x + yi$

Then $|x+yi+4| \le 3 \Rightarrow |(x+4)+yi| \le 3$

$\Rightarrow \quad \sqrt{(x+4)^2 + y^2} \le 3 \Rightarrow x^2 + 8x + 16 + y^2 \le 9$

$\Rightarrow \quad x^2 + y^2 + 8x + 7 \le 0$

Which is a circle having centre (–4, 0) and $r = \sqrt{(4)^2 - 7} = \sqrt{9} = 3$ and is on or inside the circle.

Hence, (*f*) ↔ (*vi*).

(*g*) Let $z = \dfrac{1+2i}{1-i}$

$$= \frac{1+2i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i+2i+2i^2}{1-i^2}$$

$$= \frac{1+i+2i-2}{1+1} = \frac{-1+3i}{2} = -\frac{1}{2} + \frac{3}{2}i$$

$\therefore \quad \bar{z} = -\dfrac{1}{2} - \dfrac{3}{2}i$ which lies in third quadrant.

Hence, (*g*) ↔ (*viii*).

(*h*) Given that: $z = 1 - i$

Reciprocal of $z = \dfrac{1}{z} = \dfrac{1}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{1+i}{1-i^2}$

$$= \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

Which lies in first quadrant.

Hence, (*h*) ↔ (*vii*).

Hence, the correct matches are (*a*) ↔ (*v*), (*b*) ↔ (*iii*), (*c*) ↔ (*i*), (*d*) ↔ (*iv*), (*e*) ↔ (*ii*), (*f*) ↔ (*vi*), (*g*) ↔ (*viii*), (*h*) ↔ (*vii*).

Q28. What is the conjugate of $\dfrac{2-i}{(1-2i)^2}$?

Sol. Given that $z = \dfrac{2-i}{(1-2i)^2} = \dfrac{2-i}{1+4i^2-4i} = \dfrac{2-i}{1-4-4i}$

$$= \frac{2-i}{-3-4i} = \frac{2-i}{-3-4i} \times \frac{-3+4i}{-3+4i}$$

$$= \frac{-6+8i+3i-4i^2}{(-3)^2-(4i)^2} = \frac{-6+11i+4}{9-16i^2}$$

$$= \frac{-2+11i}{9+16} = \frac{-2+11i}{25} = \frac{-2}{25} + \frac{11}{25}i$$

$\therefore \quad \bar{z} = \frac{-2}{25} - \frac{11}{25}i$

Hence, $\quad \bar{z} = \frac{-2}{25} - \frac{11}{25}i.$

Q29. If $|z_1| = |z_2|$, is it necessary that $z_1 = z_2$?

Sol. Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$

$\therefore \quad |x_1 + y_1 i| = |x_2 + y_2 i| \Rightarrow \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$

$\Rightarrow \quad x_1^2 + y_1^2 = x_2^2 + y_2^2 \Rightarrow x_1^2 = x_2^2$ and $y_1^2 = y_2^2$

$\Rightarrow \quad x_1 = \pm x_2$ and $y_1 = \pm y_2$

So $\quad z_1 = x_1 + y_1 i$ and $z_2 = \pm x_2 \pm y_2 i$

$\therefore \quad z_1 \neq z_2$

Hence, it is not necessary that $z_1 = z_2$.

Q30. If $\frac{(a^2+1)^2}{2a-i} = x + iy$, then what is the value of $x^2 + y^2$?

Sol. Given that: $\frac{(a^2+1)^2}{2a-i} = x + iy \quad \ldots(i)$

Taking conjugate on both sides

$\Rightarrow \quad \frac{(a^2+1)^2}{2a+i} = x - iy \quad \ldots(ii)$

Multiplying eqn. (*i*) and (*ii*) we have

$$\frac{(a^2+1)^2\,(a^2+1)^2}{(2a-i)(2a+i)} = x^2 + y^2 \Rightarrow \frac{(a^2+1)^4}{4a^2 - i^2} = x^2 + y^2$$

$\Rightarrow \quad \frac{(a^2+1)^4}{4a^2+1} = x^2 + y^2$

Hence, the value of $x^2 + y^2 = \frac{(a^2+1)^4}{4a^2+1}$.

Q31. Find the value of z, if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.

Sol. Given that: $|z| = 4$ and $\arg(z) = \frac{5\pi}{6} \Rightarrow \theta = \frac{5\pi}{6}$

$|z| = 4 \Rightarrow r = 4$

So Polar form of $z = r\,[\cos\theta + i\sin\theta]$

$$= 4\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$$

$$= 4\left[\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right]$$

$$= 4\left[-\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right] = 4\left[\frac{-\sqrt{3}}{2} + i.\frac{1}{2}\right]$$

$$= -2\sqrt{3} + 2i$$

Hence, $z = -2\sqrt{3} + 2i$.

Q32. Find $\left|(1+i)\frac{(2+i)}{(3+i)}\right|$.

Sol. $\left|(1+i)\frac{(2+i)}{(3+i)} \times \frac{3-i}{3-i}\right|$

$$= \left|(1+i).\frac{6-2i+3i-i^2}{9-i^2}\right| = \left|\frac{(1+i).(7+i)}{9+1}\right|$$

$$= \left|\frac{7+i+7i+i^2}{10}\right| = \left|\frac{7+8i-1}{10}\right|$$

$$= \left|\frac{6+8i}{10}\right| = \left|\frac{3}{5} + \frac{4}{5}i\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= 1$$

Hence, $\left|(1+i)\left(\frac{2+i}{3+i}\right)\right| = 1$

Q33. Find principal argument of $(1+i\sqrt{3})^2$.

Sol. Given that: $(1+i\sqrt{3})^2 = 1 + i^2.3 + 2\sqrt{3}i$

$$= 1 - 3 + 2\sqrt{3}i = -2 + 2\sqrt{3}i$$

$$\tan\alpha = \left|\frac{2\sqrt{3}}{-2}\right| \qquad \left[\because \tan\alpha = \left|\frac{\text{Img}(z)}{\text{Re}(z)}\right|\right]$$

$\Rightarrow$ $\tan\alpha = \left|-\sqrt{3}\right| = \sqrt{3}$

$\Rightarrow$ $\tan\alpha = \tan\frac{\pi}{3} \quad \therefore \alpha = \frac{\pi}{3}$

Now Re$(z) < 0$ and image $(z) > 0$.

$\therefore$ $\arg(z) = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Hence, the principal arg $= \frac{2\pi}{3}$.

Q34. Where does z lie, if $\left|\frac{z-5i}{z+5i}\right| = 1$.

Sol. Given that: $\left|\frac{z-5i}{z+5i}\right| = 1$

Let $z = x + yi$

$\therefore \quad \left|\frac{x+yi-5i}{x+yi+5i}\right| = 1 \Rightarrow \left|\frac{x+(y-5)i}{x+(y+5)i}\right| = 1$

$\Rightarrow \quad |x+(y-5)i| = |x+(y+5)i|$

$\Rightarrow \quad x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow \quad (y-5)^2 = (y+5)^2$

$\Rightarrow \quad y^2 + 25 - 10y = y^2 + 25 + 10y$

$\Rightarrow \quad 20y = 0 \Rightarrow y = 0$

Hence, z lies on x-axis i.e., real axis.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer out of the given four options in each of the Exercises from 35 to 50 (M.C.Q.)

Q35. $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for:

(*a*) $x = n\pi$ (*b*) $x = \left(n+\frac{1}{2}\right).\frac{\pi}{2}$

(*c*) $x = 0$ (*d*) No value of x

Sol. Let $z = \sin x + i\cos 2x$

$\bar{z} = \sin x - i\cos 2x$

But we are given that $\bar{z} = \cos x - i\sin 2x$

$\therefore \quad \sin x - i\cos 2x = \cos x - i\sin 2x$

Comparing the real and imaginary parts, we get

$\sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow \quad \tan x = 1$ and $\tan 2x = 1$

$\Rightarrow \quad \tan x = \tan\frac{\pi}{4}$ and $\tan 2x = \tan\frac{\pi}{4}$

$\therefore \quad x = n\pi + \frac{\pi}{4}, \; n \in I$ and $2x = n\pi + \frac{\pi}{4}$

$\Rightarrow \quad x = 2x \Rightarrow 2x - x = 0 \Rightarrow x = 0$

Hence, the correct option is (*c*).

Q36. The real value of α for which the expression $\frac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely real is:

(*a*) $(n+1)\frac{\pi}{2}$ (*b*) $(2n+1)\frac{\pi}{2}$

(*c*) $n\pi$ (*d*) None of these, where $n \in N$

Sol. Let $$z = \frac{1 - i\sin\alpha}{1 + 2i\sin\alpha} = \frac{(1 - i\sin\alpha)(1 - 2i\sin\alpha)}{(1 + 2i\sin\alpha)(1 - 2i\sin\alpha)}$$

$$= \frac{1 - 2i\sin\alpha - i\sin\alpha + 2i^2\sin^2\alpha}{(1)^2 - (2i\sin\alpha)^2}$$

$$= \frac{1 - 3i\sin\alpha - 2\sin^2\alpha}{1 - 4i^2\sin^2\alpha} = \frac{(1 - 2\sin^2\alpha) - 3i\sin\alpha}{1 + 4\sin^2\alpha}$$

$$= \frac{1 - 2\sin^2\alpha}{1 + 4\sin^2\alpha} - \frac{3\sin\alpha}{1 + 4\sin^2\alpha}.i$$

Since, z is purely real, then

$$\frac{-3\sin\alpha}{1 + 4\sin^2\alpha} = 0 \Rightarrow \sin\alpha = 0$$

So, $\alpha = n\pi,\ n \in \mathrm{N}$.

Hence, the correct option is (*c*).

Q37. If $z = x + iy$ lies in the third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant if

(*a*) $x > y > 0$ (*b*) $x < y < 0$

(*c*) $y < x < 0$ (*d*) $y > x > 0$

Sol. Given that: $z = x + iy$

If z lies in third quadrant.

So $x < 0$ and $y < 0$.

$$\bar{z} = x - iy$$

$$\frac{\bar{z}}{z} = \frac{x - iy}{x + iy} = \frac{x - iy}{x + iy} \times \frac{x - iy}{x - iy}$$

$$= \frac{x^2 + i^2y^2 - 2xyi}{x^2 - i^2y^2} = \frac{x^2 - y^2 - 2xyi}{x^2 + y^2}$$

$$= \frac{x^2 - y^2}{x^2 + y^2} - \frac{2xy}{x^2 + y^2}i$$

When z lies in third quadrant then $\frac{\bar{z}}{z}$ will also be lie in third quadrant

$$\therefore \quad \frac{x^2 - y^2}{x^2 + y^2} < 0 \quad \text{and} \quad \frac{-2xy}{x^2 + y^2} < 0$$

$$\Rightarrow \quad x^2 - y^2 < 0 \quad \text{and} \quad 2xy > 0$$

$$\Rightarrow \quad x^2 < y^2 \quad \text{and} \quad xy > 0$$

So $x < y < 0$.

Hence, the correct option is (*b*)

Q38. The value of $(z+3)(\bar{z}+3)$ is equivalent to

(*a*) $|z+3|^2$ (*b*) $|z-3|$

(*c*) z^2+3 (*d*) None of these

Sol. Given that: $(z+3)(\bar{z}+3)$

Let $z = x + yi$

So $(z+3)(\bar{z}+3) = (x+yi+3)(x-yi+3)$

$= [(x+3)+yi][(x+3)-yi]$

$= (x+3)^2 - y^2i^2 = (x+3)^2 + y^2$

$= |x+3+iy|^2 = |z+3|^2$

Hence, the correct option is (*a*).

Q39. If $\left(\dfrac{1+i}{1-i}\right)^x = 1$, then

(*a*) $x = 2n+1$ (*b*) $x = 4n$

(*c*) $x = 2n$ (*d*) $x = 4n+1$, where $n \in N$

Sol. Given that: $\left(\dfrac{1+i}{1-i}\right)^x = 1$

$$\Rightarrow \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^x = 1 \Rightarrow \left(\frac{1+i^2+2i}{1-i^2}\right)^x = 1$$

$$\Rightarrow \left(\frac{1-1+2i}{1+1}\right)^x = 1 \Rightarrow \left(\frac{2i}{2}\right)^x = 1$$

$$\Rightarrow (i)^x = (i)^{4n}$$

$$\Rightarrow x = 4n,\ n \in N$$

Hence, the correct option is (*b*).

Q40. A real value of *x* satisfies the equation

$\left(\dfrac{3-4ix}{3+4ix}\right) = \alpha - i\beta$ $(\alpha, \beta \in R)$ if $\alpha^2 + \beta^2$ is equal to

(*a*) 1 (*b*) -1 (*c*) 2 (*d*) -2

Sol. Given that: $\left(\dfrac{3-4ix}{3+4ix}\right) = \alpha - i\beta$

$$\Rightarrow \left(\frac{3-4ix}{3+4ix} \times \frac{3-4ix}{3-4ix}\right) = \alpha - i\beta$$

$$\Rightarrow \frac{9-12ix-12ix+16i^2x^2}{9-16i^2x^2} = \alpha - i\beta$$

$$\Rightarrow \frac{9-24ix-16x^2}{9+16x^2} = \alpha - i\beta$$

$$\Rightarrow \frac{9-16x^2}{9+16x^2} - \frac{24x}{9+16x^2}i = \alpha - i\beta \quad \text{...}(i)$$

$$\Rightarrow \quad \frac{9-16x^2}{9+16x^2} + \frac{24x}{9+16x^2}i = \alpha + i\beta \quad ...(ii)$$

Multiplying eqn. (i) and (ii) we get

$$\left(\frac{9-16x^2}{9+16x^2}\right)^2 + \left(\frac{24x}{9+16x^2}\right)^2 = \alpha^2 + \beta^2$$

$$\Rightarrow \quad \frac{(9-16x^2)^2 + (24x)^2}{(9+16x^2)^2} = \alpha^2 + \beta^2$$

$$\Rightarrow \quad \frac{81 + 256x^4 - 288x^2 + 576x^2}{(9+16x^2)^2} = \alpha^2 + \beta^2$$

$$\Rightarrow \quad \frac{81 + 256x^4 + 288x^2}{(9+16x^2)^2} = \alpha^2 + \beta^2$$

$$\Rightarrow \quad \frac{(9+16x^2)^2}{(9+16x^2)^2} = \alpha^2 + \beta^2$$

So, $\alpha^2 + \beta^2 = 1$

Hence, the correct option is (a).

Q41. Which of the following is correct for any two complex numbers z_1 and z_2?

(a) $|z_1 z_2| = |z_1||z_2|$ (b) $\arg(z_1 z_2) = \arg(z_1).\arg(z_2)$

(c) $|z_1 + z_2| = |z_1| + |z_2|$ (d) $|z_1 + z_2| \geq |z_1| - |z_2|$

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\therefore \quad |z_1| = r_1$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \quad |z_2| = r_2$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \,.\, r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2(\cos\theta_1 + i\sin\theta_1) \,.\, (\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$$
$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$
$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$\therefore \quad |z_1 z_2| = |z_1||z_2|$

Hence, the correct option is (a).

Q42. The point represented by the complex number $(2 - i)$ is rotated about origin through an angle $\frac{\pi}{2}$ in clockwise direction, the new position of point is

(a) $1 + 2i$ (b) $-1 - 2i$ (c) $2 + i$ (d) $-1 + 2i$

Sol. Given that: $z = 2 - i$

If z rotated through an angle of $\frac{\pi}{2}$ about the origin in clockwise direction.

Then the new position $= z.e^{-(\pi/2)}$

$$= (2 - i)\, e^{-(\pi/2)}$$

$$= (2 - i)\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$$

$$= (2 - i)\,(0 - i) = -1 - 2i$$

Hence, the correct option is (b).

Q43. If $x, y \in$ R, then $x + iy$ is a non-real complex number if

(a) $x = 0$ (b) $y = 0$ (c) $x \neq 0$ (d) $y \neq 0$

Sol. $x + yi$ is a non-real complex number if $y \neq 0$. If $x, y \in$ R.

Hence, the correct option is (d).

Q44. If $a + ib = c + id$, then

(a) $a^2 + c^2 = 0$ (b) $b^2 + c^2 = 0$

(c) $b^2 + d^2 = 0$ (d) $a^2 + b^2 = c^2 + d^2$

Sol. Given that: $a + ib = c + id$

$$\Rightarrow \quad |a + ib| = |c + id|$$

$$\Rightarrow \quad \sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$$

Squaring both sides, we get $a^2 + b^2 = c^2 + d^2$

Hence, the correct option is (d).

Q45. The complex number z which satisfies the condition $\left|\frac{i + z}{i - z}\right| = 1$ lies on

(a) circle $x^2 + y^2 = 1$ (b) the x-axis

(c) the y-axis (d) the line $x + y = 1$

Sol. Given that: $\left|\frac{i + z}{i - z}\right| = 1$

Let $z = x + yi$

$$\therefore \quad \left|\frac{i + x + yi}{i - x - yi}\right| = 1 \Rightarrow \left|\frac{x + (y + 1)i}{-x - (y - 1)i}\right| = 1$$

$$\Rightarrow \quad |x + (y + 1)i| = |-x - (y - 1)i|$$

$$\Rightarrow \quad \sqrt{x^2 + (y + 1)^2} = \sqrt{x^2 + (y - 1)^2}$$

$$\Rightarrow \quad x^2 + (y + 1)^2 = x^2 + (y - 1)^2 \Rightarrow (y + 1)^2 = (y - 1)^2$$

$$\Rightarrow \quad y^2 + 2y + 1 = y^2 - 2y + 1 \Rightarrow 2y = -2y$$

$$\Rightarrow \quad 4y = 0 \Rightarrow y = 0 \Rightarrow x\text{-axis.}$$

Hence, the correct option is (b).

Q46. If z is a complex number, then

(*a*) $|z^2| > |z|$ (*b*) $|z^2| = |z|^2$

(*c*) $|z^2| < |z|^2$ (*d*) $|z^2| \geq |z|^2$

Sol. Let $z = x + yi$

$$|z| = |x + yi| \text{ and } |z|^2 = |x + yi|^2$$

$\Rightarrow |z|^2 = x^2 + y^2$ ...(*i*)

Now $z^2 = x^2 + y^2i^2 + 2xyi$

$z^2 = x^2 - y^2 + 2xyi$

$$|z^2| = \sqrt{(x^2 - y^2)^2 + (2xy)^2} = \sqrt{x^4 + y^4 - 2x^2y^2 + 4x^2y^2}$$

$$= \sqrt{x^4 + y^4 + 2x^2y^2} = \sqrt{(x^2 + y^2)^2}$$

So $|z|^2 = x^2 + y^2 = |z|^2$

So $|z|^2 = |z^2|$

Hence, the correct option is (*b*).

Q47. $|z_1 + z_2| = |z_1| + |z_2|$ is possible if

(*a*) $z_2 = \bar{z}_1$ (*b*) $z_2 = \dfrac{1}{z_1}$

(*c*) $\arg(z_1) = \arg(z_2)$ (*d*) $|z_1| = |z_2|$

Sol. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Since $|z_1 + z_2| = |z_1| + |z_2|$

$$z_1 + z_2 = r_1\cos\theta_1 + i\,r_1\sin\theta_1 + r_2\cos\theta_2 + i\,r_2\sin\theta_2$$

$$|z_1 + z_2| = \sqrt{\begin{aligned} & r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 + 2r_1r_2\cos\theta_1\cos\theta_2 \\ & + r_1^2\sin^2\theta_1 + r_2^2\sin^2\theta_2 + 2r_1r_2\sin\theta_1\sin\theta_2 \end{aligned}}$$

$$= \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}$$

But $|z_1 + z_2| = |z_1| + |z_2|$

So $\sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)} = r_1 + r_2$

Squaring both sides, we get

$$r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1r_2$$

$\Rightarrow 2r_1r_2 - 2r_1r_2\cos(\theta_1 - \theta_2) = 0$

$\Rightarrow 1 - \cos(\theta_1 - \theta_2) = 0 \Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \theta_1 - \theta_2 = 0 \Rightarrow \theta_1 = \theta_2$

So, $\arg(z_1) = \arg(z_2)$

Hence, the correct option is (*c*).

Q48. The real value of θ for which the expression $\dfrac{1 + i\cos\theta}{1 - 2i\cos\theta}$ is a real number is (*a*) $n\pi + \frac{\pi}{4}$ (*b*) $n\pi + (-1)^n \frac{\pi}{4}$

(*c*) $2n\pi \pm \frac{\pi}{2}$ (*d*) None of these

Sol. Let
$$z = \frac{1 + i\cos\theta}{1 - 2i\cos\theta} = \frac{1 + i\cos\theta}{1 - 2i\cos\theta} \times \frac{1 + 2i\cos\theta}{1 + 2i\cos\theta}$$
$$= \frac{1 + 2i\cos\theta + i\cos\theta + 2i^2\cos^2\theta}{1 - 4i^2\cos^2\theta}$$
$$= \frac{1 + 3i\cos\theta - 2\cos^2\theta}{1 + 4\cos^2\theta}$$
$$= \frac{1 - 2\cos^2\theta}{1 + 4\cos^2\theta} + \frac{3\cos\theta}{1 + 4\cos^2\theta}i$$

If z is a real number, then
$$\frac{3\cos\theta}{1 + 4\cos^2\theta} = 0$$
$$\Rightarrow \quad 3\cos\theta = 0 \Rightarrow \cos\theta = 0$$
$$\therefore \quad \theta = (2n+1)\frac{\pi}{2}, \quad n \in \text{N}.$$

Hence, the correct option is (*c*).

Q49. The value of arg (x), when $x < 0$ is

(*a*) 0 (*b*) $\frac{\pi}{2}$

(*c*) π (*d*) None of these

Sol. Let $z = -x + 0i$ and $x < 0$
$$\therefore \quad |z| = \sqrt{(-1)^2 + (0)^2} = 1, \quad x < 0$$

Since, the point $(-x, 0)$ lies on the negative side of the real axis ($\because x < 0$).

$\therefore$ Principal argument $(z) = \pi$

Hence, the correct option is (*c*).

Q50. If $f(z) = \frac{7 - z}{1 - z^2}$, where $z = 1 + 2i$, then $|f(z)|$ is equal to

(*a*) $\frac{|z|}{2}$ (*b*) $|z|$

(*c*) $2|z|$ (*d*) None of these

Sol. Given that: $z = 1 + 2i$
$$|z| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$$

Now
$$f(z) = \frac{7 - z}{1 - z^2}$$

$$= \frac{7-(1+2i)}{1-(1+2i)^2} = \frac{7-1-2i}{1-1-4i^2-4i}$$

$$= \frac{6-2i}{4-4i} = \frac{3-i}{2-2i} = \frac{3-i}{2-2i} \times \frac{2+2i}{2+2i}$$

$$= \frac{6+6i-2i-2i^2}{4-4i^2} = \frac{6+4i+2}{4+4}$$

$$= \frac{8+4i}{8} = 1+\frac{1}{2}i$$

So $$|f(z)| = \sqrt{(1)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2} = \frac{|z|}{2}$$

Hence, the correct option is (*a*).