2.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Sol. We know that $\frac{5\pi}{6}\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\frac{13\pi}{6}\notin[0,\pi]$

$$\therefore\ \tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

$$=\tan^{-1}\left[\tan\left(\pi-\frac{\pi}{6}\right)\right]+\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]$$

$$=\tan^{-1}\left[\tan\left(-\frac{\pi}{6}\right)\right]+\cos^{-1}\left(\cos\frac{\pi}{6}\right)$$

$$=\tan^{-1}\left(-\tan\frac{\pi}{6}\right)+\cos^{-1}\left(\cos\frac{\pi}{6}\right)$$

$$=-\tan^{-1}\left(\tan\frac{\pi}{6}\right)+\cos^{-1}\left(\cos\frac{\pi}{6}\right)\qquad[\because\ \tan^{-1}(-x)=-\tan^{-1}x]$$

$$=-\frac{\pi}{6}+\frac{\pi}{6}=0$$

Hence, $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)=0$

Q2. Evaluate: $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

Sol. $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

$$=\cos\left[\pi-\cos^{-1}\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right]\quad[\because\ \cos^{-1}(-x)=\pi-\cos^{-1}x]$$

$$=\cos\left[\pi-\cancel{\frac{\pi}{6}}+\cancel{\frac{\pi}{6}}\right]=\cos\pi=-1$$

Hence, $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]=-1.$

Q3. Prove that: $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$.

Sol. L.H.S. $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)$

$$=\cot\left[\tan^{-1}(1)-2\tan^{-1}\frac{1}{3}\right] \qquad \left[\because \cot^{-1}x=\tan^{-1}\frac{1}{x}\right]$$

$$=\cot\left[\tan^{-1}(1)-\tan^{-1}\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right] \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$=\cot\left[\tan^{-1}(1)-\tan^{-1}\frac{\frac{2}{3}}{\frac{8}{9}}\right]$$

$$=\cot\left[\tan^{-1}(1)-\tan^{-1}\frac{3}{4}\right]$$

$$=\cot\left[\tan^{-1}\left(\frac{1-\frac{3}{4}}{1+1\times\frac{3}{4}}\right)\right]=\cot\left[\tan^{-1}\left(\frac{\frac{1}{4}}{\frac{7}{4}}\right)\right]$$

$$=\cot\left[\tan^{-1}\frac{1}{7}\right] \qquad \left[\because \tan^{-1}\frac{1}{x}=\cot^{-1}x\right]$$

$=\cot[\cot^{-1}(7)]=7$ R.H.S.

Hence proved.

Q4. Find the value of $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

Sol. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$

$$= -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sqrt{3}\right) + \tan^{-1}(-1) \qquad \left[\begin{array}{l} \because \tan^{-1}(-x) = -\tan^{-1} x \\ \tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right) \\ \sin\left(\frac{-\pi}{2}\right) = -1 \end{array}\right]$$

$$= -\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{-\pi}{12} \qquad \left[\because \tan^{-1}(-1) = \frac{-\pi}{4}\right]$$

Hence, $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right] = \frac{-\pi}{12}$

Q5. Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

Sol. We know that $\frac{2\pi}{3} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \quad \tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{3}\right)\right] = \tan^{-1}\left(-\tan\frac{\pi}{3}\right)$$

$$= -\tan^{-1}\left(\tan\frac{\pi}{3}\right) \quad [\because \tan^{-1}(-x) = -\tan^{-1} x]$$

$$= -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Hence, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \frac{-\pi}{3}$.

Q6. Show that: $2\tan^{-1}(-3) = \frac{-\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right)$

Sol. L.H.S. $2\tan^{-1}(-3) = -2\tan^{-1}(3)$

$$= -\cos^{-1}\left[\frac{1-(3)^2}{1+(3)^2}\right] \qquad \left[\because 2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= -\cos^{-1}\left(\frac{1-9}{1+9}\right) = -\cos^{-1}\left(\frac{-8}{10}\right)$$

$$= -\cos^{-1}\left(\frac{-4}{5}\right) = -\left[\pi - \cos^{-1}\left(\frac{4}{5}\right)\right] = -\pi + \cos^{-1}\frac{4}{5}$$

$$= -\pi + \tan^{-1}\left(\frac{3}{4}\right) \qquad \left[\because \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}\right]$$

$$= -\pi + \frac{\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right) \qquad \left[\tan^{-1} x = \frac{\pi}{2} - \cot^{-1} x\right]$$

$$= \frac{-\pi}{2} - \cot^{-1}\left(\frac{3}{4}\right)$$

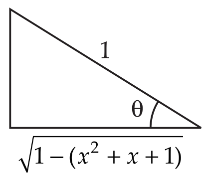
$$= \frac{-\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right) \qquad \left[\because \tan^{-1} x = \cot^{-1}\frac{1}{x}\right]$$

$$= \frac{-\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right) \text{ R.H.S.}$$

Hence proved.

Q7. Find the real solutions of the equation

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$



Sol. $$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

Let $\theta = \sin^{-1}\sqrt{x^2+x+1}$

$\therefore \quad \sin\theta = \sqrt{x^2+x+1}$

$$\Rightarrow \tan\theta = \frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}} \quad \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{x^2+x+1}}{\sqrt{-x^2-x}}\right)$$

$$\Rightarrow \sin^{-1}\sqrt{x^2+x+1} = \tan^{-1}\left(\sqrt{\frac{x^2+x+1}{-x^2-x}}\right)$$

$$\text{So, } \tan^{-1}\sqrt{x(x+1)} + \tan^{-1}\left(\sqrt{\frac{x^2+x+1}{-x^2-x}}\right) = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\sqrt{x(x+1)} + \sqrt{\dfrac{x^2+x+1}{-x(x+1)}}}{1 - \sqrt{x(x+1)} \times \sqrt{\dfrac{x^2+x+1}{-x(x+1)}}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\dfrac{x(x+1) + \sqrt{-(x^2+x+1)}}{\sqrt{x(x+1)}}}{1 - \sqrt{-(x^2+x+1)}}\right] = \frac{\pi}{2}$$

$$\Rightarrow \frac{x^2 + x - \sqrt{-(x^2+x+1)}}{\left[1-\sqrt{-(x^2+x+1)}\right]\sqrt{x^2+x}} = \tan\frac{\pi}{2} = \frac{1}{0}$$

$$\Rightarrow \left[1-\sqrt{-(x^2+x+1)}\right]\sqrt{x^2+x} = 0$$

$$\Rightarrow \left[1-\sqrt{-(x^2+x+1)}\right] = 0 \quad \text{or} \quad \sqrt{x^2+x} = 0$$

Here, $\sqrt{-(x^2+x+1)} \notin R$

$\therefore \quad \sqrt{x^2+x} = 0$

$\Rightarrow \quad x^2 + x = 0 \quad \Rightarrow \quad x(x+1) = 0$

$\Rightarrow \quad x = 0 \quad \text{or} \quad x + 1 = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = -1$

Hence the real solutions are $x = 0$ and $x = -1$.

Alternate Method:

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\sqrt{x^2+x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1}$$

$$\Rightarrow \quad \tan^{-1}\sqrt{x^2+x} = \cos^{-1}\sqrt{x^2+x+1} \quad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \cos^{-1}\left[\frac{1}{\sqrt{1+x^2+x}}\right] = \cos^{-1}\sqrt{x^2+x+1}$$

$$\left[\because \tan^{-1}x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow \quad \frac{1}{\sqrt{x^2+x+1}} = \sqrt{x^2+x+1}$$

$\Rightarrow \quad x^2 + x + 1 = 1 \quad \Rightarrow x^2 + x = 0$

$\Rightarrow \quad x(x+1) = 0 \quad \Rightarrow x = 0 \quad \text{or} \quad x + 1 = 0$

$\therefore \quad x = 0, x = -1$

Q8. Find the value of the expression

$$\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$$

Sol. $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)\right]+\cos\left[\cos^{-1}\frac{1}{\sqrt{1+\left(2\sqrt{2}\right)^2}}\right]$$

$$\left[\because \tan^{-1}x=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)\right]+\cos\left[\cos^{-1}\left(\frac{1}{3}\right)\right]$$

$$\Rightarrow \sin\left[\tan^{-1}\left(\frac{3}{4}\right)\right]+\frac{1}{3} \quad \Rightarrow \quad \sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right]+\frac{1}{3}$$

$$\Rightarrow \frac{3}{5}+\frac{1}{3} \quad \Rightarrow \frac{14}{15} \qquad \left[\because \tan^{-1}x=\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right]$$

Hence, $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)=\mathbf{\frac{14}{15}}$.

Q9. If $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$, then show that $\theta=\frac{\pi}{4}$

Sol. $2\tan^{-1}(\cos\theta)=\tan^{-1}(2\operatorname{cosec}\theta)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos\theta}{1-\cos^2\theta}\right)=\tan^{-1}(2\operatorname{cosec}\theta)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \frac{2\cos\theta}{1-\cos^2\theta}=2\operatorname{cosec}\theta \quad \Rightarrow \frac{2\cos\theta}{\sin^2\theta}=\frac{2}{\sin\theta}$$

$\Rightarrow \cos\theta\sin\theta=\sin^2\theta$

$\Rightarrow \cos\theta\sin\theta-\sin^2\theta=0 \quad \Rightarrow \sin\theta(\cos\theta-\sin\theta)=0$

$\Rightarrow \sin\theta=0$ or $\cos\theta-\sin\theta=0$

$\Rightarrow \sin\theta=0$ or $1-\tan\theta=0$

$\Rightarrow \theta=0$ or $\tan\theta=1$

$\Rightarrow \theta=0°$ or $\theta=\frac{\pi}{4}$ **Hence proved**.

Q10. Show that: $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$

Sol. L.H.S. $\cos\left(2\tan^{-1}\frac{1}{7}\right)$

$$= \cos\left[\cos^{-1}\frac{1-\frac{1}{49}}{1+\frac{1}{49}}\right] \qquad \left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}\right]$$

$$= \cos\left[\cos^{-1}\frac{48}{50}\right] = \cos\left[\cos^{-1}\frac{24}{25}\right] = \frac{24}{25}$$

R.H.S. $\sin\left[4\tan^{-1}\frac{1}{3}\right]$

$$= \sin\left[2\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}\right)\right] \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \sin\left[2\tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)\right] = \sin\left[2\tan^{-1}\frac{3}{4}\right]$$

$$= \sin\left[\sin^{-1}\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right] \qquad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= \sin\left[\sin^{-1}\frac{24}{25}\right] \Rightarrow \frac{24}{25}$$

L.H.S. = R.H.S. **Hence proved**.

Q11. Solve the following equation: $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

Sol. Given that $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$$\Rightarrow \cos\left[\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right] = \sin\left[\sin^{-1}\frac{4}{5}\right]$$

$$\left[\begin{array}{l}\because \tan^{-1}x = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) \\ \cot^{-1}x = \sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\end{array}\right]$$

$\Rightarrow \quad \dfrac{1}{\sqrt{1+x^2}} = \dfrac{4}{5}$

Squaring both sides we get,

$$\frac{1}{1+x^2} = \frac{16}{25} \Rightarrow 1+x^2 = \frac{25}{16}$$

$\Rightarrow \quad x^2 = \dfrac{25}{16} - 1 = \dfrac{9}{16} \Rightarrow x = \pm\dfrac{3}{4}$

Hence, $\quad \boldsymbol{x = \dfrac{-3}{4}, \dfrac{3}{4}}$.

LONG ANSWER TYPE QUESTIONS

Q12. Prove that: $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}x^2$

Sol. L.H.S. $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$

Put $x^2 = \cos\theta \quad \therefore \theta = \cos^{-1}x^2$

$\Rightarrow \tan^{-1}\left[\dfrac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right]$

$\Rightarrow \tan^{-1}\left[\dfrac{\sqrt{2\cos^2\theta/2}+\sqrt{2\sin^2\theta/2}}{\sqrt{2\cos^2\theta/2}-\sqrt{2\sin^2\theta/2}}\right] \quad \left[\begin{array}{l}\because 1+\cos\theta = 2\cos^2\theta/2 \\ \quad 1-\cos\theta = 2\sin^2\theta/2\end{array}\right]$

$\Rightarrow \tan^{-1}\left[\dfrac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right]$

$\Rightarrow \tan^{-1}\left[\dfrac{1+\tan\theta/2}{1-\tan\theta/2}\right]$ [Dividing the Nr. and Den. by cos θ/2]

$\Rightarrow \tan\left[\tan\left(\dfrac{\pi}{4} \quad \dfrac{\theta}{2}\right)\right] \quad \left[\because \dfrac{1+\tan\theta}{1-\tan\theta} = \tan\left(\dfrac{\pi}{4}+\theta\right)\right]$

$\Rightarrow \dfrac{\pi}{4}+\dfrac{\theta}{2} \quad \Rightarrow \dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2$ R.H.S. [Putting $\theta = \cos^{-1}x^2$]

Hence proved.

Q13. Find the simplified form of $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$, where $x \in \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right]$

Sol. Given that $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$

Put $\frac{3}{5} = \cos y$

$\therefore \quad \sqrt{1-\cos^2 y} = \sin y \Rightarrow \sqrt{1-\frac{9}{25}} = \sin y \Rightarrow \frac{4}{5} = \sin y$

$$\therefore \cos^{-1}\left[\frac{3}{5}\cos x + \frac{4}{5}\sin x\right] = \cos^{-1}[\cos y \cos x + \sin y \sin x]$$

$$= \cos^{-1}[\cos(y-x)] = y - x$$

$$= \tan^{-1}\frac{4}{3} - x \qquad \left[\tan y = \frac{\sin y}{\cos y} = \frac{4}{3}\right]$$

Q14. Prove that: $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$

Sol. L.H.S. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$

Using $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\left[\frac{8}{17}.\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}.\sqrt{1-\left(\frac{8}{17}\right)^2}\right]$$

$$= \sin^{-1}\left[\frac{8}{17}.\sqrt{1-\frac{9}{25}} + \frac{3}{5}.\sqrt{1-\frac{64}{289}}\right]$$

$$= \sin^{-1}\left[\frac{8}{17}.\sqrt{\frac{16}{25}} + \frac{3}{5}.\sqrt{\frac{225}{289}}\right]$$

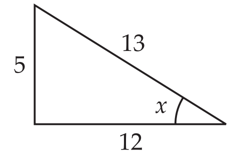
$$= \sin^{-1}\left[\frac{8}{17}.\frac{4}{5} + \frac{3}{5}.\frac{15}{17}\right] = \sin^{-1}\left[\frac{32}{85} + \frac{45}{85}\right]$$

$= \sin^{-1}\frac{77}{85}$ R.H.S. **Hence proved.**

Q15. Show that: $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$

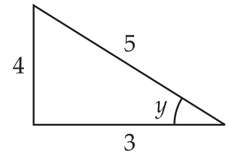
Sol. Let $\sin^{-1}\frac{5}{13}=x \quad \Rightarrow \quad \sin x=\frac{5}{13}$

$\Rightarrow \quad \tan x=\frac{5}{12}$



Let $\cos^{-1}\frac{3}{5}=y \quad \Rightarrow \quad \cos y=\frac{3}{5}$

$\Rightarrow \quad \tan y=\frac{4}{3}$



Now $\quad \tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$

$$\Rightarrow \quad \tan(x+y)=\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}=\frac{\frac{15+48}{36}}{\frac{36-20}{36}}=\frac{63}{16}$$

$$\Rightarrow \quad x+y=\tan^{-1}\frac{63}{16}$$

$\therefore \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$ **Hence proved.**

Q16. Prove that: $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\sin^{-1}\frac{1}{\sqrt{5}}$

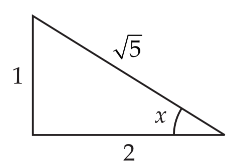
Sol. $\quad \tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\tan^{-1}\left[\dfrac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right]$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right]=\tan^{-1}\left[\frac{17}{34}\right]$$

Let $\quad \tan^{-1}\left[\frac{17}{34}\right]=x$

$\therefore \quad \tan x=\frac{17}{34}=\frac{1}{2}$



$$\sin x = \frac{1}{\sqrt{5}}$$

$$\therefore \qquad \tan^{-1}\frac{1}{2} = \sin^{-1}\frac{1}{\sqrt{5}} \text{ R.H.S.}$$

Hence, $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$

Q17. Find the value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

Sol. $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$

$$\Rightarrow 2.\left(2\tan^{-1}\frac{1}{5}\right) - \tan^{-1}\frac{1}{239}$$

$$\Rightarrow 2\left[\tan^{-1}\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right] - \tan^{-1}\frac{1}{239} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow 2\tan^{-1}\frac{5}{12} - \tan^{-1}\frac{1}{239} \quad \Rightarrow \tan^{-1}\left(\frac{2\times\frac{5}{12}}{1-\frac{25}{144}}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right] \qquad \left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{120\times 239 - 119}{119\times 239 + 120}\right] \Rightarrow \tan^{-1}\left[\frac{28680-119}{28441+120}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{28561}{28561}\right] = \tan^{-1}(1) = \frac{\pi}{4}$$

Q18. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

Sol. To prove that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

L.H.S. Let $\frac{1}{2}\sin^{-1}\frac{3}{4} = \tan^{-1}\theta \qquad [\therefore \tan(\tan^{-1}\theta) = \theta]$

$$\Rightarrow \quad \sin^{-1}\frac{3}{4} = 2\tan^{-1}\theta \Rightarrow \sin^{-1}\frac{3}{4} = \sin^{-1}\left(\frac{2\theta}{1+\theta^2}\right)$$

$$\left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$\Rightarrow \quad \frac{2\theta}{1+\theta^2} = \frac{3}{4} \Rightarrow 3 + 3\theta^2 = 8\theta$$

$$\Rightarrow \quad 3\theta^2 - 8\theta + 3 = 0$$

$$\Rightarrow \quad \theta = \frac{-(-8) \pm \sqrt{(-8)^2 - 4\times3\times3}}{2\times3}$$

$$= \frac{8\pm\sqrt{64-36}}{6} = \frac{8\pm\sqrt{28}}{6} = \frac{8\pm2\sqrt{7}}{6} = \frac{2\left(4\pm\sqrt{7}\right)}{6}$$

$$\Rightarrow \quad \theta = \frac{4\pm\sqrt{7}}{3}$$

$$\therefore \quad \theta = \frac{4+\sqrt{7}}{3} \text{ or } \frac{4-\sqrt{7}}{3}$$

$$\theta = \frac{4+\sqrt{7}}{3} \text{ is ignored.}$$

Because $\frac{-\pi}{2} \le \sin^{-1}\frac{3}{4} \le \frac{\pi}{2}$

$$\Rightarrow \frac{-\pi}{4} \le \frac{1}{2}\sin^{-1}\frac{3}{4} \le \frac{\pi}{4}$$

$$\Rightarrow \tan\left(\frac{-\pi}{4}\right) \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow -1 \le \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) \le 1$$

Hence, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

Q19. If $a_1, a_2, a_3, ..., a_n$ is an arithmetic progression with common difference d, then evaluate the following expression

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \tan^{-1}\left(\frac{d}{1+a_3a_4}\right) + ... \right.$$
$$\left. ... + \tan^{-1}\left(\frac{d}{1+a_{n-1}.a_n}\right)\right]$$

Sol. If $a_1, a_2, a_3, ..., a_n$ are the terms of an arithmetic progression
$\therefore\ d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3$ ....

$$\therefore\ \tan\left[\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{a_4-a_3}{1+a_3a_4}\right)+\ldots\right.$$
$$\left.\ldots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}.a_n}\right)\right]$$

$$\Rightarrow \tan\left[(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)+(\tan^{-1}a_4-\tan^{-1}a_3)\right.$$
$$\left.+\ldots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})\right]$$
$$\left[\because \tan^{-1}\frac{x-y}{1+xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$\Rightarrow \tan\left[\tan^{-1}a_2-\tan^{-1}a_1+\tan^{-1}a_3-\tan^{-1}a_2+\tan^{-1}a_4-\tan^{-1}a_3\right.$$
$$\left.+\ldots+\tan^{-1}a_n-\tan^{-1}a_{n-1}\right]$$

$$\Rightarrow \tan\left[\tan^{-1}a_n-\tan^{-1}a_1\right]$$

$$\Rightarrow \tan\left[\tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)\right] \Rightarrow \frac{a_n-a_1}{1+a_1a_n} \qquad [\because \tan(\tan^{-1}x)=x]$$

OBJECTIVE TYPE QUESTIONS

Choose the correct answers from the given four options in each of the Exercises from 20 to 37 (M.C.Q.).

Q20. Which of the following is the principal value branch of $\cos^{-1}x$?

(*a*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (*b*) $(0, \pi)$

(*c*) $[0, \pi]$ (*d*) $(0, \pi)-\left\{\frac{\pi}{2}\right\}$

Sol. Principal value branch of $\cos^{-1}x$ is $[0, \pi]$. Hence the correct answer is (*c*).

Q21. Which of the following is the principal value branch of $\operatorname{cosec}^{-1}x$?

(*a*) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ (*b*) $[0, \pi]-\left\{\frac{\pi}{2}\right\}$

(*c*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ (*d*) $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$

Sol. Principal value branch of $\operatorname{cosec}^{-1}x$ is

$\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-\{0\}$ as $\operatorname{cosec}^{-1}(0)=\infty$ (not defined).

Hence, the correct answer is (*d*).

Q22. If $3 \tan^{-1} x + \cot^{-1} x = \pi$, then x equals

(*a*) 0 (*b*) 1 (*c*) -1 (*d*) $\frac{1}{2}$

Sol. Given that $3 \tan^{-1} x + \cot^{-1} x = \pi$

$\Rightarrow 2 \tan^{-1} x + \tan^{-1} x + \cot^{-1} x = \pi$

$\Rightarrow \quad 2 \tan^{-1} x + \frac{\pi}{2} = \pi \qquad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$

$\Rightarrow \quad 2 \tan^{-1} x = \pi - \frac{\pi}{2} \quad \Rightarrow 2 \tan^{-1} x = \frac{\pi}{2}$

$\Rightarrow \quad \tan^{-1} x = \frac{\pi}{4} \quad \Rightarrow \tan^{-1} x = \tan^{-1}(1)$

$\therefore \quad x = 1$

Hence, the correct answer is (*b*).

Q23. The value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ is

(*a*) $\frac{3\pi}{5}$ (*b*) $\frac{-7\pi}{5}$ (*c*) $\frac{\pi}{10}$ (*d*) $\frac{-\pi}{10}$

Sol. $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right] = \sin^{-1}\left[\cos\left(6\pi + \frac{3\pi}{5}\right)\right]$

$= \sin^{-1}\left[\cos \frac{3\pi}{5}\right] \qquad [\because \cos(2n\pi + x) = \cos x]$

$= \sin^{-1}\left[\cos\left(\frac{\pi}{2} + \frac{\pi}{10}\right)\right]$

$= \sin^{-1}\left[-\sin\left(\frac{\pi}{10}\right)\right] \quad \left[\because \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta\right]$

$= \sin^{-1}\left[\sin\left(\frac{-\pi}{10}\right)\right] = \frac{-\pi}{10}$

Hence, the correct answer is (*d*).

Q24. The domain of the function $\cos^{-1}(2x - 1)$ is

(*a*) [0, 1] (*b*) [– 1, 1] (*c*) (– 1, 1) (*d*) [0, π]

Sol. The given function is $\cos^{-1}(2x - 1)$

Let $\quad f(x) = \cos^{-1}(2x - 1)$

$-1 \le 2x - 1 \le 1 \Rightarrow -1 + 1 \le 2x \le 1 + 1$

$0 \le 2x \le 2 \quad \Rightarrow 0 \le x \le 1$

$\therefore$ domain of the given function is [0, 1].

Hence, the correct answer is (*a*)

Q25. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(*a*) [1, 2] (*b*) [– 1, 1] (*c*) [0, 1] (*d*) None of these

Sol. Let $f(x) = \sin^{-1}\sqrt{x-1}$

$\because \sqrt{x-1} \geq 0$ and $-1 \leq \sqrt{x-1} \leq 1$

$\Rightarrow 0 \leq x - 1 \leq 1 \quad \Rightarrow \quad 1 \leq x \leq 2 \quad \Rightarrow \quad x \in [1, 2]$

Hence, the correct answer is (*a*).

Q26. If $\cos\left[\sin^{-1}\frac{2}{5} + \cos^{-1} x\right] = 0,$ then x is equal to

(*a*) $\frac{1}{5}$ (*b*) $\frac{2}{5}$ (*c*) 0 (*d*) 1

Sol. Given that $\cos\left[\sin^{-1}\frac{2}{5} + \cos^{-1} x\right] = 0$

$$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1} x = \cos^{-1}(0)$$

$$\Rightarrow \quad \sin^{-1}\frac{2}{5} + \cos^{-1} x = \frac{\pi}{2} \Rightarrow \sin^{-1}\frac{2}{5} = \frac{\pi}{2} - \cos^{-1} x$$

$$\Rightarrow \quad \sin^{-1}\frac{2}{5} = \sin^{-1} x \quad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad x = \frac{2}{5}$$

Hence, the correct answer is (*b*).

Q27. The value of sin [2 $\tan^{-1}$ (0.75)] is equal to

(*a*) 0.75 (*b*) 1.5 (*c*) 0.96 (*d*) sin 1.5

Sol. Given that sin [2 $\tan^{-1}$ (0.75)]

$$= \sin\left[2\tan^{-1}\frac{3}{4}\right]$$

$$= \sin\left[\sin^{-1}\frac{2 \times \frac{3}{4}}{1 + \frac{9}{16}}\right] \quad \left[\because 2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= \sin\left[\sin^{-1}\frac{\frac{3}{2}}{\frac{25}{16}}\right] = \sin\left[\sin^{-1}\frac{24}{25}\right]$$

$$= \sin[\sin^{-1}(0.96)]$$

$= 0.96$

Hence, the correct answer is (*c*).

Q28. The value of $\cos^{-1}\left(\cos\dfrac{3\pi}{2}\right)$ is equal to

(*a*) $\dfrac{\pi}{2}$ (*b*) $\dfrac{3\pi}{2}$ (*c*) $\dfrac{5\pi}{2}$ (*d*) $\dfrac{7\pi}{2}$

Sol. $\cos^{-1}\left(\cos\dfrac{\pi}{2}\right) \neq \dfrac{3\pi}{2} \quad \because \dfrac{3\pi}{2} \notin [0, \pi]$

$\Rightarrow \cos^{-1}\left[\cos\left(\pi + \dfrac{\pi}{2}\right)\right] \Rightarrow \cos^{-1}\left[-\cos\dfrac{\pi}{2}\right] \Rightarrow \cos^{-1}[0] = \dfrac{\pi}{2}$

Hence, the correct answer is (*a*).

Q29. The value of expression $2\sec^{-2} 2 + \sin^{-1}\left(\dfrac{1}{2}\right)$ is

(*a*) $\dfrac{\pi}{6}$ (*b*) $\dfrac{5\pi}{6}$ (*c*) $\dfrac{7\pi}{6}$ (*d*) 1

Sol. $2\sec^{-1} 2 + \sin^{-1}\dfrac{1}{2} = 2\sec^{-1}\left(\sec\dfrac{\pi}{3}\right) + \sin^{-1}\left(\sin\dfrac{\pi}{6}\right)$

$$= 2.\frac{\pi}{3} + \frac{\pi}{6} = \frac{2\pi}{3} + \frac{\pi}{6} = \frac{5\pi}{6}$$

Hence, the correct answer is (*b*).

Q30. If $\tan^{-1} x + \tan^{-1} y = \dfrac{4\pi}{5}$, then $\cot^{-1} x + \cot^{-1} y$ equals

(*a*) $\dfrac{\pi}{5}$ (*b*) $\dfrac{2\pi}{5}$ (*c*) $\dfrac{3\pi}{5}$ (*d*) π

Sol. Given that $\tan^{-1} x + \tan^{-1} y = \dfrac{4\pi}{5}$

$$\Rightarrow \frac{\pi}{2} - \cot^{-1} x + \frac{\pi}{2} - \cot^{-1} y = \frac{4\pi}{5} \quad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$$

$$\Rightarrow \pi - (\cot^{-1} x + \cot^{-1} y) = \frac{4\pi}{5}$$

$$\Rightarrow \cot^{-1} x + \cot^{-1} y = \pi - \frac{4\pi}{5}$$

$$\Rightarrow \cot^{-1} x + \cot^{-1} y = \frac{\pi}{5}$$

Hence, the correct answer is (*a*).

Q31. If $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right) + \cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$,

where $a, x \in]\,0, 1$, then the value of x is

(*a*) 0 (*b*) $\dfrac{a}{2}$ (*c*) a (*d*) $\dfrac{2a}{1-a^2}$

Sol. $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\Rightarrow \quad 2\tan^{-1} a + 2\tan^{-1} a = 2\tan^{-1} x$

$\left[\because 2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}\right]$

$\Rightarrow \quad 4\tan^{-1} a = 2\tan^{-1} x \quad \Rightarrow 2\tan^{-1} a = \tan^{-1} x$

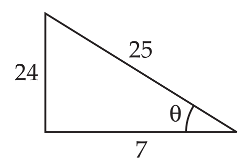
$\Rightarrow \quad \tan^{-1}\frac{2a}{1-a^2} = \tan^{-1} x \quad \Rightarrow x = \frac{2a}{1-a^2}$

Hence, the correct answer is (*d*).

Q32. The value of $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$ is

(*a*) $\frac{25}{24}$ (*b*) $\frac{25}{7}$ (*c*) $\frac{24}{25}$ (*d*) $\frac{7}{24}$

Sol. We have, $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$



Let $\quad \cos^{-1}\frac{7}{25} = \theta$

$\therefore \quad \cos\theta = \frac{7}{25} \Rightarrow \cot\theta = \frac{7}{24}$

$\therefore \cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right] = \cot\left[\cot^{-1}\left(\frac{7}{24}\right)\right] = \frac{}{24}$

Hence, the correct answer is (*d*).

Q33. The value of expression $\tan\left[\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right]$ is

(*a*) $2+\sqrt{5}$ (*b*) $\sqrt{5}-2$ (*c*) $\frac{\sqrt{5}+2}{2}$ (*d*) $5+\sqrt{2}$

Sol. We have, $\tan\left[\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right]$

Let $\quad \theta = \frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}$

$\Rightarrow \quad 2\theta = \cos^{-1}\frac{2}{\sqrt{5}} \Rightarrow \cos 2\theta = \frac{2}{\sqrt{5}}$

$\Rightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{2}{\sqrt{5}} \quad \left[\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

$\Rightarrow \quad 2+2\tan^2\theta = \sqrt{5}-\sqrt{5}\tan^2\theta$

$\Rightarrow \sqrt{5}\tan^2\theta + 2\tan^2\theta = \sqrt{5}-2 \Rightarrow \left(\sqrt{5}+2\right)\tan^2\theta = \sqrt{5}-2$

$$\Rightarrow \qquad \tan^2\theta = \frac{\sqrt{5}-2}{\sqrt{5}+2}$$

$$\Rightarrow \qquad \tan^2\theta = \frac{(\sqrt{5}-2)(\sqrt{5}-2)}{(\sqrt{5}+2)(\sqrt{5}-2)} \Rightarrow \tan^2\theta = \frac{(\sqrt{5}-2)^2}{5-4}$$

$$\Rightarrow \qquad \tan\theta = \pm(\sqrt{5}-2)$$

$$\Rightarrow \qquad \tan\theta = \sqrt{5}-2, \quad \left[-(\sqrt{5}-2) \text{ is not required}\right]$$

Hence, the correct answer is (*b*).

Q34. If $|x| \le 1$, then $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

(*a*) $4\tan^{-1}x$ (*b*) 0 (*c*) $\frac{\pi}{2}$ (*d*) π

Sol. Here, we have $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$= 2\tan^{-1}x + 2\tan^{-1}x \qquad \left[\because 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right]$$

$$= 4\tan^{-1}x$$

Hence, the correct answer is (*a*).

Q35. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ equals

(*a*) 0 (*b*) 1 (*c*) 6 (*d*) 12

Sol. We have $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$

$\Rightarrow \quad \cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = \pi + \pi + \pi$

$\Rightarrow \quad \cos^{-1}\alpha = \pi, \cos^{-1}\beta = \pi$ and $\cos^{-1}\gamma = \pi$

$\Rightarrow \quad \alpha = \cos\pi, \beta = \cos\pi$ and $\gamma = \cos\pi$

$\therefore \quad \alpha = -1, \beta = -1$ and $\gamma = -1$

Which gives $\alpha = \beta = \gamma = -1$

So $\quad \alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$

$\Rightarrow \quad (-1)(-1-1) + (-1)(-1-1) + (-1)(-1-1)$

$\Rightarrow \quad (-1)(-2) + (-1)(-2) + (-1)(-2) \Rightarrow 2+2+2 \Rightarrow 6$

Hence, the correct answer is (*c*).

Q36. The number of real solutions of the equation

$\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2}, \pi\right]$ is (*a*) 0 (*b*) 1 (*c*) 2 (*d*) infinite

Sol. We have $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$

$\Rightarrow \quad \sqrt{2\cos^2 x} = \sqrt{2}x \quad [\because \cos^{-1}(\cos x) = x]$

$\Rightarrow \quad \sqrt{2}\cos x = \sqrt{2}x \Rightarrow \cos x = x$

Which does not satisfy for any value of x.

Hence, the correct answer is (*d*).

Q37. If $\cos^{-1} x > \sin^{-1} x$, then

(*a*) $\frac{1}{\sqrt{2}} < x \le 1$ (*b*) $0 \le x < \frac{1}{\sqrt{2}}$

(*c*) $-1 \le x < \frac{1}{\sqrt{2}}$ (*d*) $x > 0$

Sol. Here, given that $\cos^{-1} x > \sin^{-1} x$

$\Rightarrow \quad \sin[\cos^{-1} x] > x$

$\Rightarrow \quad \sin\left[\sin^{-1}\sqrt{1-x^2}\right] > x \quad \Rightarrow \quad \sqrt{1-x^2} > x$

$\Rightarrow \quad x < \sqrt{1-x^2} \quad \Rightarrow x^2 < 1 - x^2 \Rightarrow 2x^2 < 1$

$\Rightarrow \quad x^2 < \frac{1}{2} \Rightarrow x < \pm\frac{1}{\sqrt{2}}$

We know that $-1 \le x \le 1$

So $-1 \le x < \frac{1}{\sqrt{2}}$.

Hence, the correct answer is (c).

Fill in the Blanks in each of the Exercises 38 to 48.

Q38. The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is ....

Sol. Let $\cos^{-1}\left(-\frac{1}{2}\right) = x \quad \Rightarrow \quad \cos x = -\frac{1}{2}$

$\Rightarrow \quad \cos x = \cos\left(-\frac{\pi}{3}\right) \quad \Rightarrow \cos x = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$

$\therefore \quad x = \frac{2\pi}{3} \in [0, \pi]$

Hence, Principal value of $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

Q39. The value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is ....

Sol. $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) \ne \frac{3\pi}{5}$ as $\frac{3\pi}{5} \notin \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

So $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\sin\left(\pi - \frac{2\pi}{5}\right)$

$$= \sin^{-1}\sin\left(\frac{2\pi}{5}\right) = \frac{2\pi}{5} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Hence, the value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}$

Q40. If $\cos\left(\tan^{-1}x + \cot^{-1}\sqrt{3}\right) = 0,$ then value of x is ....

Sol. Given that

$$\cos\left[\tan^{-1}x + \cot^{-1}\sqrt{3}\right] = 0$$

$$\Rightarrow \quad \tan^{-1}x + \cot^{-1}\sqrt{3} = \cos^{-1}(0)$$

$$\Rightarrow \quad \tan^{-1}x + \cot^{-1}\sqrt{3} = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{2} - \cot^{-1}\sqrt{3} \qquad \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \tan^{-1}x = \tan^{-1}\sqrt{3} \Rightarrow x = \sqrt{3}$$

Hence, the value of x is $\sqrt{3}$.

Q41. The set of values of $\sec^{-1}\left(\frac{1}{2}\right)$ is ....

Sol. Let $\quad \sec^{-1}\left(\frac{1}{2}\right) = x \quad \Rightarrow \sec x = \frac{1}{2}$

Since, the domain of $\sec^{-1}x$ is $R - \{-1, 1\}$ and $\frac{1}{2} \notin R - \{-1, 1\}$.

Hence, $\sec^{-1}\left(\frac{1}{2}\right)$ has no set of values.

Q42. The principal value of $\tan^{-1}\sqrt{3}$ is ....

Sol. $\tan^{-1}\sqrt{3} = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Hence the principal value of $\tan^{-1}\sqrt{3}$ is $\frac{\pi}{3}$.

Q43. The value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is ....

Sol.
$$\cos^{-1}\left(\cos\frac{14\pi}{3}\right) = \cos^{-1}\left[\cos\left(5\pi - \frac{\pi}{3}\right)\right]$$

$$= \cos^{-1}\left[\cos\left(\frac{-\pi}{3}\right)\right] = \cos^{-1}\left[\cos\left(\pi - \frac{\pi}{3}\right)\right]$$

$$= \cos^{-1}\left[\cos\frac{2\pi}{3}\right] = \frac{2\pi}{3} \in [0, \pi]$$

Hence, the value of $\cos^{-1}\left[\cos\frac{14\pi}{3}\right] = \frac{2\pi}{3}$.

Q44. The value of $\cos(\sin^{-1}x + \cos^{-1}x)$, $|x| \leq 1$ is ....

Sol. $\cos[\sin^{-1}x + \cos^{-1}x] = \cos\dfrac{\pi}{2} = 0 \qquad \left[\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}\right]$

Hence, the value of $\cos(\sin^{-1}x + \cos^{-1}x) = 0$.

Q45. The value of expression $\tan\left(\dfrac{\sin^{-1}x + \cos^{-1}x}{2}\right)$, when $x = \dfrac{\sqrt{3}}{2}$ is ....

Sol. $\tan\left(\dfrac{\sin^{-1}x + \cos^{-1}x}{2}\right) = \tan\left(\dfrac{\pi}{4}\right) = 1 \quad \left[\because \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}\right]$

Hence, the value of the given expression is 1.

Q46. If $y = 2\tan^{-1}x + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ for all x, then ..... $< y <$ ......

Sol.

$$y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad y = 2\tan^{-1}x + 2\tan^{-1}x$$

$$\Rightarrow \quad y = 4\tan^{-1}x \qquad \left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x\right]$$

Now $\dfrac{-\pi}{2} < \tan^{-1}x < \dfrac{\pi}{2}$

$\Rightarrow \quad -4 \times \dfrac{\pi}{2} < 4\tan^{-1}x < 4 \times \dfrac{\pi}{2} \quad \Rightarrow -2\pi < y < 2\pi$

Hence, the value of y is $(-2\pi, 2\pi)$.

Q47. The result $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$ is true when value of xy is ....

Sol. The given result is true when $xy > -1$.

Q48. The value of $\cot^{-1}(-x)$ for all $x \in R$ in terms of $\cot^{-1}x$ is ....

Sol. $\cot^{-1}(-x) = \pi - \cot^{-1}x, x \in R \qquad [\because \text{ as } \cot^{-1}(-x) = \pi - \cot^{-1}x]$

State True or False for the Statement in Each of the Exercises 49 to 55.

Q49. All trigonometric functions have inverse over their respective domains.

Sol. False.

We know that all inverse trigonometric functions are restricted over their domains.

Q50. The value of expression $(\cos^{-1}x)^2$ is equal to $\sec^2 x$.

Sol. False.

We know that $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right) \neq \sec x$

So $(\cos^{-1} x)^2 \neq \sec^2 x$

Q51. The domain of trigonometric functions can be restricted to any one of their branch (not necessarily principal value) in order to obtain their inverse functions.

Sol. True.

We know that all trigonometric functions are restricted over their domains to obtain their inverse functions.

Q52. The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

Sol. True.

Q53. The graph of inverse trigonometric function can be obtained from the graph of their corresponding trigonometric function by interchanging x and y *axes*.

Sol. True.

We know that the domain and range are interchanged in the graph of inverse trigonometric functions to that of their corresponding trigonometric functions.

Q54. The minimum value of n for which $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$, $n \in \mathrm{N}$ is valid is 5.

Sol. False.

Given that $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$

$\Rightarrow \quad \frac{n}{\pi} > \tan\frac{\pi}{4} \quad \Rightarrow \quad \frac{n}{\pi} > 1$

$\Rightarrow \quad n > \pi \quad \Rightarrow \quad n > 3.14$

Hence, the value of n is 4.

Q55. The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ is $\frac{\pi}{3}$.

Sol. True.

$$\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right] = \sin^{-1}\left[\cos\left(\sin^{-1}\sin\frac{\pi}{6}\right)\right]$$

$$\sin^{-1}\left[\cos\frac{\pi}{6}\right] = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$