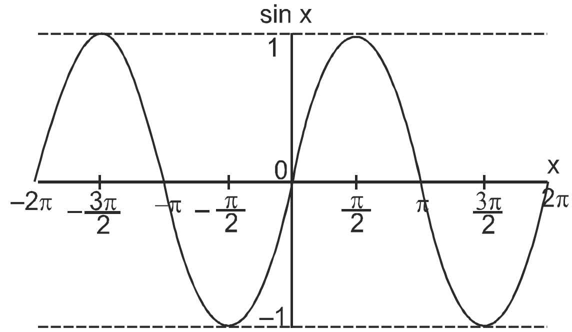
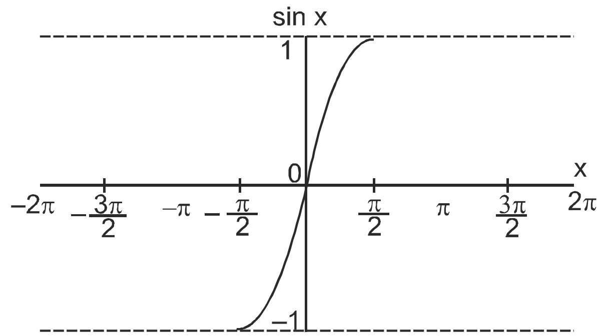
INVERSE TRIGONOMETRY

INTRODUCTION

It is known that trigonometric functions are not bijective function, and thus they do not have inverse function. However, trigonometric functions can be converted to bijective functions by restricting their domains. For example, function f(x) = sin x is not a one-to-one function from $-\infty$ to $+\infty$ a and has no inverse function. This can be seen on the figure. However, it is a bijective function from $-\pi/2$ to $+\pi/2$, and thus f(x) has inverse function in this range.

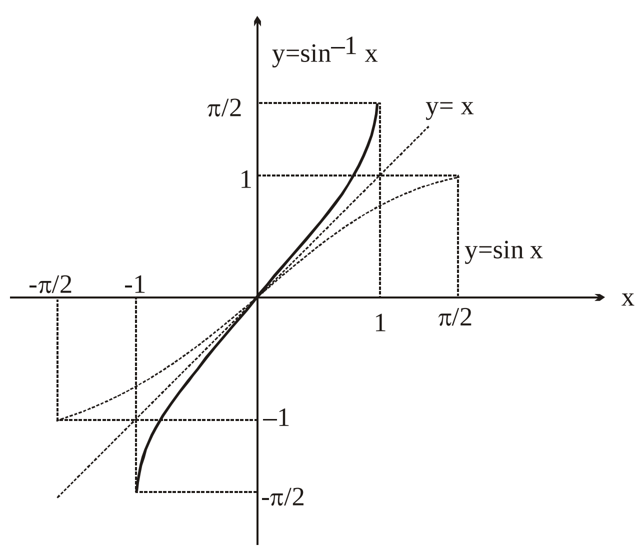


Hence, to find inverse of a trigonometric function we restrict their domain to convert trigonometric functions into one-one function according to graph given below.

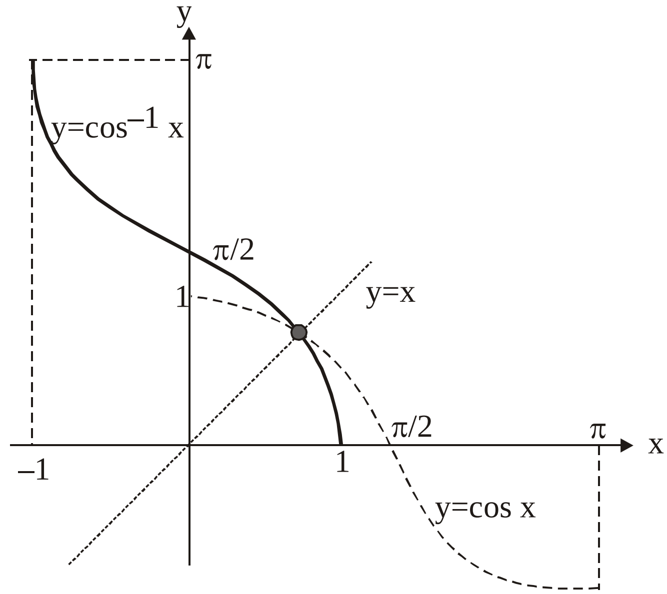


GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

$\sin\theta = x \Rightarrow \theta = \sin^{-1} x$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x \in [-1, 1]$

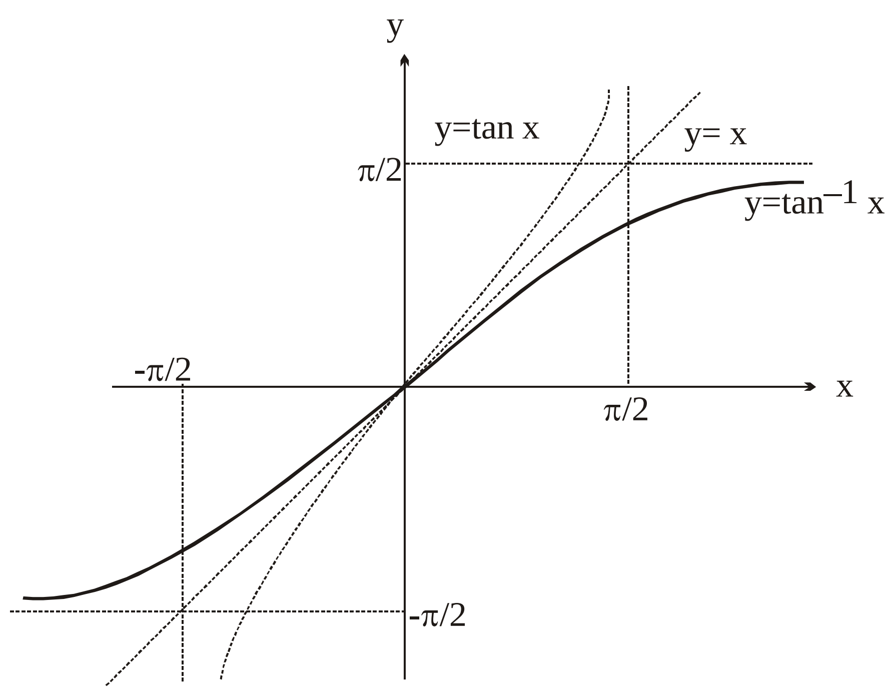


$\cos\theta = x \Rightarrow \theta = \cos^{-1} x$, where $\theta \in [0, \pi]$ and $x \in [-1, 1]$



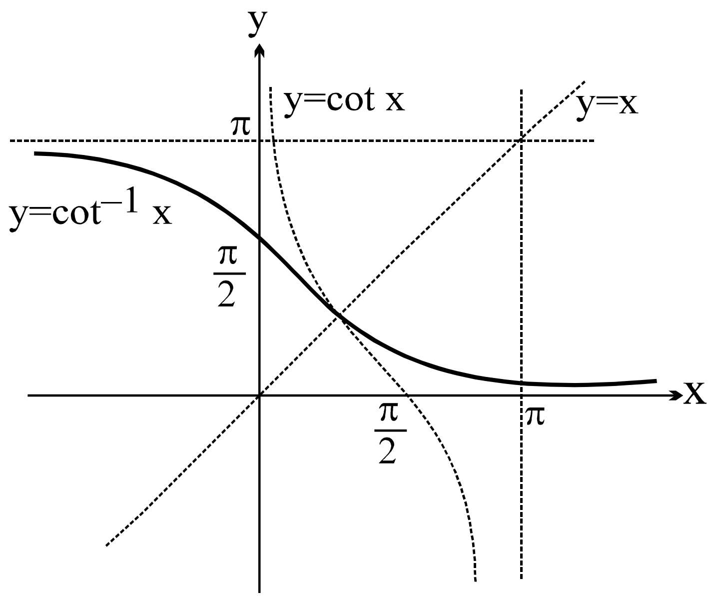
Note : $\cos^{-1} x$ is a decreasing function in [–1, 1]

tan θ = x $\Rightarrow$ $\theta = \tan^{-1} x$ where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and x $\in (-\infty, \infty)$



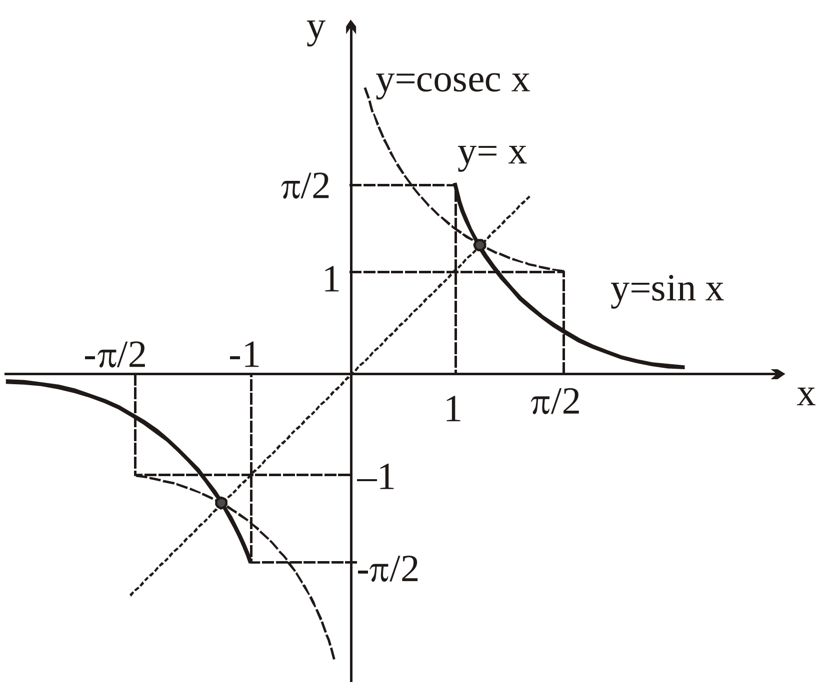
Note : $\tan^{-1} x$ is an increasing function in R.

cot θ = x $\Rightarrow$ $\theta = \cot^{-1} x$ where $\theta \in (0, \pi)$ and x $\in (-\infty, \infty)$



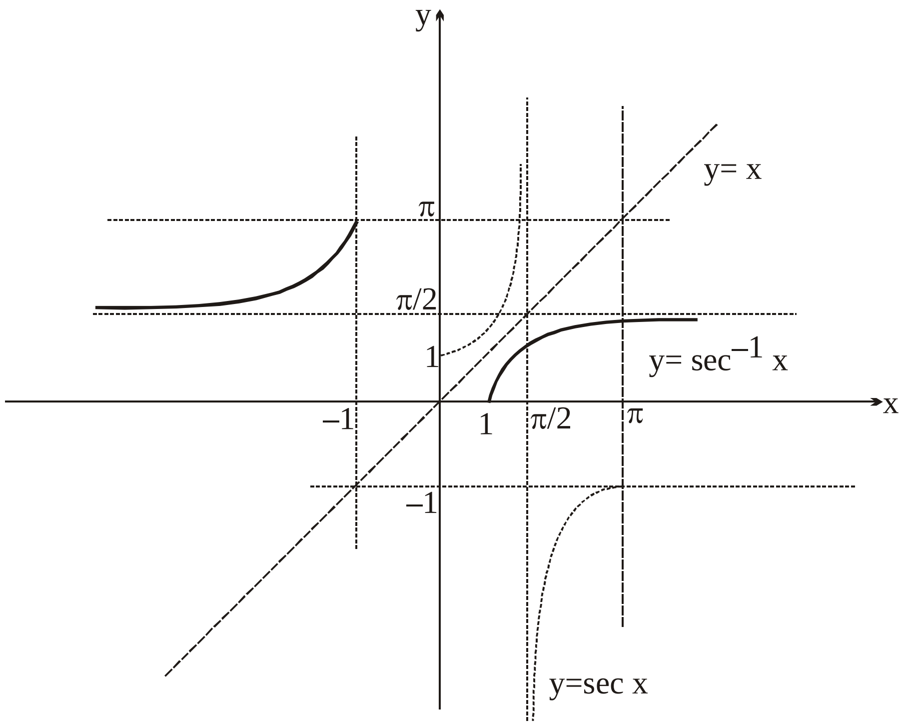
Note : $\tan^{-1} x$ is an decreasing function in R.

cosec θ = x $\Rightarrow$ θ = cosec^{-1} x where $\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ and x $\in$ $(-\infty, -1] \cup [1, \infty)$



Note : cosec^{-1} x is a decreasing function in $(-\infty, -1]$. It also decreases in $[1, \infty)$

sec θ = x $\Rightarrow$ θ = $\sec^{-1}$x here $\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ and x $\in$ $(-\infty, -1] \cup [1, \infty)$



Note : $\sec^{-1} x$ is an increasing function in $(-\infty, -1]$. It also increases in $[1, \infty)$

Principal Values & Domains of Inverse Trigonometric/Circular Functions:

Function	Domain	Range
☞ $y = \sin^{-1} x$ where	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
☞ $y = \cos^{-1} x$ where	$-1 \le x \le 1$	$0 \le y \le \pi$
☞ $y = \tan^{-1} x$ where	$x \in R$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
☞ $y = \text{cosec}^{-1} x$ where	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$
☞ $y = \sec^{-1} x$ where	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi$; $y \ne \frac{\pi}{2}$
☞ $y = \cot^{-1} x$ where	$x \in R$	$0 < y < \pi$

TYPE - 1 PROBLEMS BASED ON WRITING RANGE OTHER THAN PRINCIPAL BRANCH

ILLUSTRATIONS

1. Write the range of one branch of $\sin^{-1} x$, other than principal branch.

Ans. $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

2. Can range of $\cos^{-1} x$, be taken as $[2\pi, 3\pi]$ other than the principal branch.

Ans. Yes

PRACTICE QUESTIONS

1. Write the range of one branch of $\cos^{-1}x$, other than Principal Branch.

2. Can range of $\sin^{-1}x$ be taken as $\left[\frac{3\pi}{2},\frac{5\pi}{2}\right]$ other than the principal branch?

3. Write the principal value branch of $\tan^{-1}x$.

4. Is the function $\sec^{-1}x$, is defined at x = 0.

5. Write the range of one branch of $\cot^{-1}x$, other than Principal Branch.

6. Write the range of two more branches of $\text{cosec}^{-1}x$, other than Principal Branch.

ANSWERS

1. $[\pi, 2\pi]$ **2.** Yes **3.** $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ **4.** No **5.** $(\pi, 2\pi)$ **6.** $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]-\{\pi\}$, $\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right]-\{-\pi\}$

TYPE - 2 PROBLEMS BASED ON PRINCIPAL BRANCH OF INVERSE TRIGONOMETRIC FUNCTIONS

ILLUSTRATIONS

1. $\text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$ as $\frac{\pi}{3}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

2. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\left(\cos\frac{\pi}{6}\right)=\frac{\pi}{6}$ as $\frac{\pi}{6}\in[0,\pi]$ $\left(\because\frac{3\pi}{5}\in[0,\pi]\right)$

3. $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\tan^{-1}\left(\tan\frac{\pi}{6}\right)=\frac{\pi}{6}$ as $\frac{\pi}{6}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

4. $\cot–1\left(\frac{1}{\sqrt{3}}\right)=\cos–1\left(\cot\frac{\pi}{3}\right)=\frac{\pi}{3}$ as $\frac{\pi}{3}\in(0,\pi)$

5. $\sec^{-1}(2)=\sec^{-1}\left(\sec\frac{\pi}{3}\right)=\frac{\pi}{3}$ as $\frac{\pi}{3}\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$.

6. $\text{cosec}^{-1}(\sqrt{2})=\text{cosec}^{-1}\left(\text{cosec}\frac{\pi}{4}\right)=\frac{\pi}{4}$ as $\frac{\pi}{4}\in\left[-\frac{\pi}{4},0\right)\cup\left(0,\frac{\pi}{2}\right]$.

PRACTICE QUESTIONS

1. Find the principal value of the following:

i. $\sin^{-1}\left(\frac{1}{2}\right)$ **ii.** $\cos^{-1}\left(\frac{1}{2}\right)$ **iii.** $\tan^{-1}(\sqrt{3})$ **iv.** $\text{cosec}^{-1}(-2)$ **v.** $\tan^{-1}(-1)$

vi. $\sin^{-1}\left(-\frac{1}{2}\right)$ **vii.** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **viii.** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ **ix.** $\text{cosec}^{-1}(-1)$ **x.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

xi. $\cot^{-1}(-\sqrt{3})$ **xii.** $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ **xiii.** $\sec^{-1}(-\sqrt{2})$ **xiv.** $\cos^{-1}(-1)$

ANSWERS

1. **i.** $\pi/6$ **ii.** $\pi/3$ **iii.** $-\pi/3$ **vi.** $-\pi/6$ **v.** $-\pi/4$ **vi.** $-\pi/6$ **vii.** $-\pi/3$
viii. $-\pi/6$ **ix.** $-\pi/2$ **x.** $5\pi/6$ **xi.** $5\pi/6$ **xii.** $2\pi/3$ **xiii.** $3\pi/4$ **xiv.** π

TYPE-3 PROBLEMS BASED ON SIMPLYFYING SUM OR DIFFERENCE OF TWO OR MORE INVERSE FUNCTIONS USING PRINCIPAL BRANCH

ILLUSTRATIONS

1. Using the principal value, evaluate the following: $\tan^{-1} 1 + \sin^{-1}\left(-\frac{1}{2}\right)$

Sol. $\tan^{-1} 1 + \sin^{-1}\left(-\frac{1}{2}\right)$ Range of the principal value of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and range of the principal value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let $\tan^{-1} 1 = \theta$, then $\tan \theta = 1 = \tan \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Again, let $\sin^{-1}\left(-\frac{1}{2}\right) = \phi$, then $\sin \phi = -\frac{1}{2} = -\sin \frac{\pi}{6}$. Now, $\sin \phi = -\frac{1}{2} = -\sin \frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \phi = -\frac{\pi}{6} \Rightarrow \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

So, $\tan^{-1} 1 + \sin^{-1}\left(-\frac{1}{2}\right) = \theta + \phi = \frac{\pi}{4} + \left(-\frac{\pi}{6}\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi - 2\pi}{12} = \frac{\pi}{12}$.

2. Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.

Sol. Range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$ and range of the pricipal value branch of sin| is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Let $\cos^{-1}\left(\frac{1}{2}\right) = \theta$, the $\cos \theta = \frac{1}{2} = \cos \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3} \in [0, \pi]$

Again, let $\sin^{-1}\left(\frac{1}{2}\right) = \phi$, then $\sin \phi = \frac{1}{2} = \sin \frac{\pi}{6}$

$\therefore \phi = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Now, $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \theta + 2\phi = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$.

Hence, the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

PRACTICE QUESTIONS

1. Evaluate following by using principle values

i. $\tan^{-1}(-1) + \sec^{-1}(\sqrt{2}) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **ii.** $\cos^{-1}\left(-\frac{1}{2}\right) + 2\sin^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}(-\sqrt{3})$

iii. $\cot^{-1}(-\sqrt{3}) + \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) + \cot^{-1}(-1)$

ANSWERS

1. **i.** $\pi/3$ **ii.** 0 **iii.** $7\pi/3$

TYPE - 4 PROBLEMS BASED ON $f^{-1} \circ f$

(i) $\sin^{-1}(\sin x) = x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(ii) $\cos^{-1}(\cos x) = x;\quad 0 \le x \le \pi$

(iii) $\tan^{-1}(\tan x) = x;\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $\cot^{-1}(\cot x) = x;\quad 0 < x < \pi$

(v) $\sec^{-1}(\sec x) = x;\ 0 \le x \le \pi,\ x \ne \frac{\pi}{2}$

(vi) $\text{cosec}^{-1}(\text{cosec}\, x) = x;\quad x \ne 0, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

ILLUSTRATIONS

1. $\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \frac{2\pi}{5}.\quad \left(\because\ \frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

2. $\sin^{-1}\left(\tan\frac{\pi}{3}\right) = \sin^{-1}\sqrt{3}$ is not defined as $\sqrt{3} \notin [-1,1]$ = domain of $\sin^{-1} x$.

3. $\cos^{-1}=\left(\cos\frac{9\pi}{8}\right) = \cos^{-1}\left(\cos\left(\pi+\frac{\pi}{8}\right)\right) = \cos^{-1}\left(-\cos\frac{\pi}{8}\right) = \cos^{-1}\left(\cos\left(\pi-\frac{\pi}{8}\right)\right) = \cos^{-1}\left(\cos\frac{7\pi}{8}\right) = \frac{7\pi}{8}$

$\left(\because\ \frac{7\pi}{8} \in [0,\pi] \text{ but } \frac{9\pi}{8} \in [0,\pi]\right)$

4. $\cos\left(\cos^{-1}\left(\frac{11}{10}\right)\right)$ is meaningless as $\cos^{-1}\left(\frac{11}{10}\right)$ is not defined. $\left(\frac{11}{10} \notin [-1,1] = \text{domain of } \cos^{-1} x\right)$

5. $\sin^{-1}\left(\sin\left(-\frac{\pi}{8}\right)\right) = \sin^{-1}\left(-\sin\frac{\pi}{8}\right) = \sin^{-1}\left(\sin\frac{\pi}{8}\right) = -\frac{\pi}{8}$

6. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left(\tan\left(\pi+\frac{\pi}{6}\right)\right) = \tan^{-1}\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}.\quad \left(\text{Note that } \frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ but } \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

7. $\cot^{-1}\left(\cot\left(-\frac{\pi}{7}\right)\right) = \cot^{-1}\left(-\cot\left(\frac{\pi}{7}\right)\right) \cot^{-1}\left(\cot\left(\pi-\frac{\pi}{7}\right)\right) = \cot^{-1}\left(\frac{6\pi}{7}\right) = \frac{6\pi}{7}$.

PRACTICE QUESTIONS

1. Find the principal value for the following :

i. $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$ **ii.** $\tan^{-1}\left(\tan\frac{\pi}{3}\right)$ **iii.** $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ **iv.** $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$

v. $\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$ **vi.** $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ **vii.** $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ **viii.** $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

ix. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$ **x.** $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ **xi.** $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$ **xii.** $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

xiii. $\cos^{-1}\left(\cos\left(\frac{-\pi}{6}\right)\right)$ **xiv.** $\cos^{-1}\left(\cos\left(\frac{-\pi}{4}\right)\right)$ **xv.** $\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$ **xvi.** $\cot^{-1}\left(\cos\left(-\frac{\pi}{2}\right)\right)$

xvii. $\sec^{-1}\left(\sqrt{2}\sin\left(\frac{-\pi}{2}\right)\right)$ **xviii.** $\cos^{-1}\left(\frac{1}{2}\sin\left(\frac{-\pi}{2}\right)\right)$

2. Find $\sin^{-1}(\sin x)$ in the following cases

i. $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ **ii.** $x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ **iii.** $x \in \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$

3. Find $\cos^{-1}(\cos x)$ in the following cases

i. $x \in (-\pi, 0)$ **ii.** $x \in (\pi, 2\pi)$ **iii.** $x \in (2\pi, 3\pi)$

4. Find $\tan^{-1}(\tan x)$ in the following cases

i. $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ **ii.** $x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ **iii.** $x \in \left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$

ANSWERS

1. **i.** $\pi/6$ **ii.** $\pi/3$ **iii.** $2\pi/3$ **iv.** $2\pi/5$ **v.** $\pi/4$ **vi.** $\pi/6$ **vii.** $\pi/6$ **viii.** $5\pi/6$ **ix.** $3\pi/4$ **x.** $2\pi/3$ **xi.** $-\pi/6$ **xii.** $-\pi/3$ **xiii.** $5\pi/6$ **xiv.** $3\pi/4$ **xv.** $-\pi/4$ **xvi.** $-\pi/2$ **xvii.** $3\pi/4$ **xviii.** $2\pi/3$

2. **i.** $\pi - x$ **ii.** $-2\pi + x$ **iii.** $-\pi - x$ **3.** **i.** $-x$ **ii.** $2\pi - x$ **iii.** $-2\pi + x$ **4.** **i.** $x - \pi$ **ii.** $-2\pi + x$ **iii.** $\pi + x$

TYPE - 5 PROBLEMS BASED ON SIMPLFYING INVERSE TRIGONOMETRIC FUNCTIONS APPEARING AS ANGLE OF TRIGONOMETRIC FUNCTIONS

PROPERTY-1

(i) $\sin^{-1}(-x) = -\sin^{-1}x,\ -1 \le x \le 1$

Proof : Let $\sin^{-1}(-x) = \theta$ $\qquad \because -1 \le x \le 1$

$\Rightarrow \sin\theta = -x$ $\qquad -1 \le -\sin\theta \le 1$

$\Rightarrow x = -\sin\theta = \sin(-\theta)$ $\qquad -1 \le \sin\theta \le 1$

$\Rightarrow \sin^{-1}x = -\theta$ $\qquad -\pi/2 \le \theta \le \pi/2$

$\Rightarrow \theta = -\sin^{-1}x$ $\qquad -\pi/2 \le -\theta \le \pi/2$

$\Rightarrow \sin^{-1}(-x) = -\sin^{-1}x$ $\qquad$ Hence Proved.

(ii) $\tan^{-1}(-x) = -\tan^{-1}x,\ x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x,\ -1 \le x \le 1$

Proof : Let $\cos^{-1}(-x) = \theta$ $\qquad \because -1 \le x \le 1$

$\Rightarrow -x = \cos\theta$ $\qquad -1 \le -\cos\theta \le 1 \Rightarrow -1 \le \cos\theta \le 1$

$\Rightarrow x = -\cos\theta = \cos(\pi-\theta)$ $\qquad 0 \le \theta \le \pi$

$\Rightarrow \cos^{-1}x = \pi-\theta$ $\qquad -\pi \le -\theta \le 0$

$\Rightarrow \theta = \pi - \cos^{-1}x$ $\qquad \pi-\pi \le \pi-\theta \le \pi + 0$

$\Rightarrow \cos^{-1}(-x) = \pi - \cos^{-1}x$ $\qquad 0 \le \pi-\theta \le \pi$ $\qquad$ Hence Proved.

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x,\quad x \in R$

PROPERTY-2

(i) $\text{cosec}^{-1}(x) = \sin^{-1}(1/x),\ |x| \ge 1$

Proof : Let $\text{cosec}^{-1}x = \theta$

$\Rightarrow 1/x = \sin\theta$ $\qquad \Rightarrow \sin^{-1}(1/x) = \sin^{-1}(\sin\theta)$

$= \theta$ (as $\theta \in [-\pi/2, \pi/2]$) $\qquad = \text{cosec}^{-1}x$

(ii) $\sec^{-1}x = \cos^{-1}\{1/x\},\quad |x| \ge 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$

PROPERTY-3

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $-1 \le x \le 1$

Proof : Let A = $\sin^{-1}x$ and B = $\cos^{-1}x$

$\Rightarrow$ sin A = x and cos B = x $\Rightarrow$ sin A = cos B

$\Rightarrow$ sin A = sin (π/2 – B)

$\Rightarrow$ A = π/2 – B, because A and π/2 – B $\in$ [–π/2, π/2] $\Rightarrow$ A + B = π/2.

Similarly, we can prove

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in R$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$, $|x| \ge 1$

PROPERTY-4

(i) $\sin(\cos^{-1}x) = \cos(\sin^{-1}x) = \sqrt{1-x^2}$, $-1 \le x \le 1$

(ii) $\tan(\cot^{-1}x) = \cot(\tan^{-1}x) = \frac{1}{x}$, $x \in R$, $x \ne 0$

(iii) $\operatorname{cosec}(\sec^{-1}x) = \sec(\operatorname{cosec}^{-1}x) = \frac{|x|}{\sqrt{x^2-1}}$, $|x| > 1$

ILLUSTRATIONS

1. **a.** $\sin\left(\cos^{-1}\left(\frac{1}{5}\right)\right) = \sin\left(\sin^{-1}\sqrt{1-\left(\frac{1}{5}\right)^2}\right) = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5}$

and $\cos\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \cos\left(\cos^{-1}\left(\sqrt{1-\left(\frac{2}{3}\right)^2}\right)\right) = \frac{\sqrt{5}}{3}$.

b. $\sin^{-1}\left(\frac{3}{4}\right) = \cos^{-1}\sqrt{1-\left(\frac{3}{4}\right)^2} = \cos^{-1}\left(\frac{\sqrt{7}}{4}\right)$

and $\cos^{-1}\left(\frac{15}{17}\right) = \sin^{-1}\sqrt{1-\left(\frac{15}{17}\right)^2} = \sin^{-1}\left(\frac{\sqrt{289-225}}{17}\right) = \sin^{-1}\left(\frac{8}{17}\right)$

2. **Evalaute:**

a. $\sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$

b. $\cos\left(2\cos^{-1}\left(\frac{2}{5}\right)\right)$

Sol. **a.** $\sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$ = sin 2θ, where θ = $\sin^{-1}\left(\frac{3}{5}\right)$

= 2 sinθ cosθ = 2 $\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

= $2\times\frac{3}{5}\sqrt{1-\left(\frac{3}{5}\right)^2}$ $\left(\because \cos(\sin^{-1}x)=\sqrt{1-x^2} \text{ for } |x|\le 1\right)$ = $2\times\frac{3}{5}\times\frac{4}{5}=\frac{24}{25}$.

b. cos $\left(2\cos^{-1}\left(\frac{2}{5}\right)\right)$ = cos 2θ, where θ = $\cos^{-1}\left(\frac{2}{5}\right)$ = 2 cos² θ – 1, cos θ = $\frac{2}{5}$

= $2\left(\frac{2}{5}\right)^2-1=\frac{8}{25}-1=\frac{8-25}{25}=-\frac{17}{25}$

3. Prove that:

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$

b. $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)=x-\frac{\pi}{4}, \frac{\pi}{4}\le x\le\frac{5\pi}{4}$

Sol. **a.** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\sin^{-1}\left(\sin\frac{\pi}{3}\right)=-\frac{\pi}{3}$

b. Given expression = $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), \frac{\pi}{4}\le x\le\frac{5\pi}{4}$

= $\cos^{-1}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)=\cos^{-1}\left(\sin\frac{\pi}{4}\sin x+\cos\frac{\pi}{4}\cos x\right), \ 0\le x-\frac{\pi}{4}\le\pi$

= $\cos^{-1}\left(\cos\left(x-\frac{\pi}{4}\right)\right), x-\frac{\pi}{4}\in[0,\pi]=x-\frac{\pi}{4}$

4. Find the value of **a.** $\cos^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)+\operatorname{cosec}^{-1}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$, 1 > x ≥ 0. **b.** cos (sec⁻¹ x + cosec⁻¹ x), | x | ≥ 1.

Sol. **a.** $\cos^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)+\operatorname{cosec}^{-1}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$, 1 > x ≥ 0. = $\cos^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)+\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

$\left(\because \operatorname{cosec}^{-1}t=\sin^{-1}\left(\frac{1}{t}\right), |t|\ge 1\right)$ = $\frac{\pi}{2}$ $\left(\because \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2} |x|\le 1\right)$

b. $\cos(\sec^{-1}x+\operatorname{cosec}^{-1}x)=\cos\left(\frac{\pi}{2}\right)=0$.

5. Prove that:

a. $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right), |x| \le \frac{1}{\sqrt{2}}$

b. $2\cos^{-1}x = \cos^{-1}(2x^2 - 1), 1 \ge x \ge 0.$

c. $2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$, for $\frac{1}{\sqrt{2}} \le x \le 1.$

Sol. a. Let $\sin^{-1}x = \theta$ so that $x = \sin\theta$, $\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

When $|x| \le \frac{1}{\sqrt{2}}$, then we have $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \Rightarrow -\frac{1}{\sqrt{2}} \le \sin\theta \le \frac{1}{\sqrt{2}}$

$\Rightarrow -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \quad \left(\because -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right) \quad \Rightarrow -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

Now, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right) = \sin^{-1}(2\sin\theta\cos\theta)$ $\left(\text{For } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \cos\theta \ge 0\right)$

$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}x$, as desired.

b. Let $\cos^{-1}x = \theta$ so that $x = \cos\theta$, $0 \le \theta \le \pi$.

When $1 \ge x \ge 0$, then we have $\cos\theta \ge 0 \quad \Rightarrow 0 \le \theta \le \frac{\pi}{2} \quad (\because 0 \le \theta \le \pi) \quad \Rightarrow 0 \le 2 \le \pi.$

Now, $\cos^{-1}(2x^2 - 1) = \cos^{-1}(2\cos^2\theta - 1) = \cos^{-1}(\cos 2\theta) = 2\cos^{-1}x$, as desired.

c. Let $\cos^{-1}x = \theta$, then $x = \cos\theta$ and $\theta \in [0, \pi]$

When $\frac{1}{\sqrt{2}} \le x \le 1$, then $\frac{1}{\sqrt{2}} \le \cos\theta \le 1 \quad \Rightarrow 0 \le \theta \le \frac{\pi}{4} \quad \Rightarrow 0 \le 2\pi \le \frac{\pi}{2}$

Hence $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin(\cos^{-1}x)\cos\theta = 2\left(\sqrt{1-x^2}\right)x$

$\Rightarrow \sin 2\theta = 2x\sqrt{1-x^2} \quad \Rightarrow 2\theta = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \quad \left(\because 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$\Rightarrow 2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

PRACTICE QUESTIONS

1. a. Find the value of: $\sin^{-1}\left(-\frac{1}{2}\right)$

b. Find the value of: $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

c. Find the value of: $\tan\left(\cos^{-1}\left(\frac{8}{17}\right)\right)$

2. Find the value of :

a. $\cos\left(\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{4}\right)$

b. $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right)$

3. **a.** Find the values of (i) $\tan^{-1}(\tan 5)$ (ii) $\sin\left(2\cot^{-1}\left(-\frac{5}{12}\right)\right)$.

b. Prove that: $\cos\left(\sin^{-1}\left(\frac{3}{5}\right)+\cot^{-1}\left(\frac{3}{2}\right)\right)=\frac{6}{5\sqrt{13}}$.

4. (i) $\sin\left(2\sin^{-1}\left(-\frac{4}{5}\right)\right)$ (ii) $\sin\left(2\cos^{-1}\left(-\frac{3}{5}\right)\right)$ (iii) $\sin\left(3\sin^{-1}\left(\frac{2}{5}\right)\right)$ (iv) $\cos\left(\sin^{-1}\frac{1}{4}+\cos^{-1}\frac{3}{4}\right)$

5. **a.** Prove that: $\cos^{-1}\left(\cos\left(\frac{5\pi}{3}\right)\right)=\frac{\pi}{3}$.

b. Simplify the following :

i. $\sin^{-1}\left[\frac{\sin x+\cos x}{\sqrt{2}}\right]$, $\frac{-\pi}{4} < x < \frac{\pi}{4}$

ii. $\cos^{-1}\left[\frac{3}{5}\cos x + \frac{4}{5}\sin x\right]$

iii. $\sin^{-1}\left[\frac{5}{13}\cos x + \frac{12}{13}\sin x\right]$

6. **a.** Prove that:

(i) $2\cos^{-1} x = 2\pi - \cos^{-1}(2x^2-1)$ for $-1 \le x \le 0$.

(ii) $2\sin^{-1} x = \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ for $\frac{1}{\sqrt{2}} \le x \le 1$.

(iii) $2\sin^{-1} x = -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ for $-1 \le x \le -\frac{1}{\sqrt{2}}$.

b. (i) Show that: $\sec^{-1}\left(\frac{1}{2x^2-1}\right) = 2\cos^{-1} x$, $0 \le x \le 1$, $x \ne \frac{1}{\sqrt{2}}$.

(ii) $\sec(\tan^{-1} x) = \sqrt{x^2+1}$ for all $x \in R$

(iii) $\text{cosec}(\cot^{-1} x) = \sqrt{x^2+1}$ for all $x \in R$

(iv) $\sec^{-1} x = \tan^{-1}\sqrt{x^2+1}$ for $x \ge 1$

(v) $\tan^{-1} x = \sec^{-1}\sqrt{1+x^2}$ for $x \ge 0$.

ANSWERS

1. **a.** $\frac{3\pi}{4}$ **b.** $\frac{5\pi}{6}$ **c.** $\frac{15}{9}$

2. **a.** $\frac{1}{2}\sin^{-1}(x/a)$ **b.** $\frac{1}{2}\tan^{-1}(ax)$

3. **(a)** (i) $5-2\pi$. (ii) $-\frac{120}{169}$ **(b)** $\frac{6}{5\sqrt{13}}=\text{R.H.S.}$

4. **(i)** $-\frac{24}{25}$ **(ii)** $-\frac{24}{25}$ **(iii)** $\frac{118}{125}$ **(iv)** $\frac{3\sqrt{15}-\sqrt{7}}{16}$

5. **b. i.** $x+\pi/4$ **ii.** $\sin^{-1}x - \sin^{-1}\sqrt{x}$ **iii.** $x+\tan^{-1} 5/12$

TYPE - 6 PROBLEMS BASED ON SIMPLYFYING INVERSE TRIGONMOETRIC FUNCTION WHEN ANGLE IS TRIGONMOETRIC FUNCTION

ILLUSTRATIONS

1. Write the given function in the simplest form: $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{3\pi}{2} < x < \frac{\pi}{2}$.

Sol. $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$

$= \tan^{-1}\left(\frac{\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}\right) = \tan^{-1}\left(\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$

$\left(\text{Note that } -\frac{3\pi}{2} < x < \frac{\pi}{2} \Leftrightarrow -\frac{3\pi}{4} < \frac{\pi}{2} < \frac{\pi}{4} \Leftrightarrow -\frac{\pi}{2} < \frac{\pi}{4} + \frac{x}{2} < \frac{\pi}{2}\right)$ $= \tan^{-1}\left(\tan\left(\frac{x}{4} + \frac{x}{2}\right)\right) = \frac{\pi}{4} + \frac{x}{2}$.

2. Write the given function in the simplest form $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ $x \in \left(0, \frac{\pi}{2}\right)$

Sol. $y = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ i.e $y = \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2} - x\right)}{1+\cos\left(\frac{\pi}{2} - x\right)}\right)$ i,e $y = \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{2} - x\right)/2\cos\left(\frac{\pi}{2} - x\right)/2}{2\cos^2\left(\frac{\pi}{2} - x\right)/2}\right)$

$y = \tan^{-1}\left(\frac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right)$ $y = \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right)$

$y = \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$ i.e $y = \frac{\pi}{4} - \frac{x}{2}$ *For* $0 < x < \frac{\pi}{2}, -\frac{\pi}{2} < -x < 0 \Rightarrow -\frac{\pi}{4} < -\frac{x}{2} < 0 \Rightarrow 0 < \frac{\pi}{4} - \frac{x}{2} < \frac{\pi}{4}$

PRACTICE QUESTIONS

1. Simplify the following :

i. $\tan^{-1}\left[\frac{1-\cos x}{\sin x}\right], \ 0 < x < \pi$

ii. $\tan^{-1}\left[\frac{1+\cos x}{\sin x}\right], \ 0 < x < \frac{\pi}{2}$

iii. $\tan^{-1}\left[\frac{\sin x}{1+\cos x}\right]$ $x \in \left(0, \frac{\pi}{2}\right)$

iv. $\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$ $x \in \left(0, \frac{\pi}{2}\right)$

v. $\tan^{-1}(\sec x + \tan x)$ $x \in \left(0, \frac{\pi}{2}\right)$

vi. $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right], x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

vii. $\cot^{-1}\left[\frac{\cos x}{1-\sin x}\right], x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

viii. $\tan^{-1}\left[\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right]$ $0 < x < \frac{\pi}{2}$

ix. $\tan^{-1}\left[\sqrt{\frac{1-\cos x}{1+\cos x}}\right]$, $0 < x < \pi$

x. $\tan^{-1}\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$ $x \in \left(0, \frac{\pi}{2}\right)$

xi. $\tan^{-1}\left[\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right], 0 < x < \frac{\pi}{2}$

xii. $\tan^{-1}\left[\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right], x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

xiii. $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], 0 < x < \frac{\pi}{2}$

xiv. $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right], \frac{\pi}{2} < x < \pi$

xv. $\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right], \frac{-\pi}{4} < x < \frac{\pi}{4}$

xvi. $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$ $x \in (0, \pi)$

xvii. $y = \sin^{-1}\left\{\frac{\sin x+\cos x}{\sqrt{2}}\right\}$

xviii. $y = \cos^{-1}\left\{\frac{\sin x+\cos x}{\sqrt{2}}\right\}$

xix. $y = \sin^{-1}\left\{\frac{\sqrt{3}\sin x+\cos x}{2}\right\}$

xx. $y = \cos^{-1}\left(\frac{2\cos x-3\sin x}{\sqrt{13}}\right)$

ANSWERS

1. **i.** $\frac{x}{2}$ **ii.** $\frac{\pi}{4}-\frac{x}{2}$ **iii.** $\frac{\pi}{2}$ **iv & v.** $\frac{\pi}{4}+\frac{x}{2}$ **vi.** $\frac{\pi}{4}-\frac{x}{2}$

vii. $\frac{\pi}{4}-\frac{x}{2}$ **viii.** x **ix.** x/2 **x.** $\frac{\pi}{4}-\frac{x}{2}$ **xi.** $\frac{\pi}{4}+\frac{x}{2}$

xii. $\frac{\pi}{4}-\frac{x}{2}$ **xiii.** $\frac{x}{2}$ **xiv.** $\frac{\pi}{2}-\frac{x}{2}$ **xv.** $\frac{\pi}{4}-x$ **xvi.** $\frac{\pi}{4}-x, \frac{5\pi}{4}-x$

xvii. $\frac{\pi}{4}+x$ **xviii.** $\frac{\pi}{4}-x$ or $x-\frac{\pi}{4}$ **xix.** $\frac{\pi}{6}+x$ **xx.** $x+\tan^{-1}\frac{3}{2}$

TYPE - 7 PROBLEMS BASED ON SIMPLYING INVERSE TRIGONOMETRIC FUNCTIONS USING SUBSTITUTION

ILLUSTRATIONS

1. Prove that: $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, -1 \le x \le 1$

Sol. Let $\cos^{-1} x = \theta$ so that $x = \cos\theta$ and $0 \le \theta \le \pi$.

$$\therefore \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right) = \tan^{-1}\left(\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right)$$

($\because$ $1+\cos\theta = 2\cos^2(\theta/2)$ and $1-\cos\theta = 2\sin^2(\theta/2)$)

$$= \tan^{-1}\left(\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right) = \frac{\pi}{4}-\frac{\theta}{2} \quad \left(0 \le \theta \le \pi \Leftrightarrow \frac{\pi}{4} \ge \frac{\pi}{4}-\frac{\theta}{2} \ge -\frac{\pi}{4}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x.$$

PRACTICE QUESTIONS

1. Simplify the following :

i. $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, x $\neq$ 0.

ii. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right), x \neq 0$

iii. $\tan^{-1}\left(\frac{\sqrt{1+x}-1}{\sqrt{x}}\right), x \neq 0$

iv. $\tan^{-1}\left(\frac{\sqrt{1+x}+1}{\sqrt{x}}\right), x \neq 0$

v. $\tan^{-1}\left(\frac{\sqrt{1+x^{2n}}-1}{x^n}\right), x \neq 0$

vi. y = $\tan^{-1}\left(\sqrt{1+x^2}-x\right)$

vii. y = $\cot^{-1}\left(\sqrt{1+x^2}+x\right)$

viii. y = $\tan^{-1}\left(\sqrt{1+x^{2n}}+x^n\right)$

ix. $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

x. $y = \tan^{-1}\left(\frac{2a^x}{1-a^{2x}}\right)$

xi. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

xii. $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

xiii. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

xiv. $y = \cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)$

xv. $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

xvi. $y = \sin^{-1}\left(3x-4x^3\right)$

xvii. $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

xviii. $y = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

xix. $y = \sin^{-1}\left(\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right), \frac{-1}{\sqrt{2}} < x < \frac{-1}{\sqrt{2}}$

xx. $y = \cos^{-1}\left(\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right), \frac{-1}{\sqrt{2}} < x < \frac{-1}{\sqrt{2}}$

xxi. y = $\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)$

xxii. $y = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

xxiii. $y = \tan^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right\}$

xxiv. $y = \cot^{-1}\left\{\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\}$

xxv. $y = \sin^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right)$

xxvi. $y = \cos^{-1}\left(1-2x^2\right)$

xxvii. $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

xxviii. $y = \sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$

2. Simply the following :

i. $y = \tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$, –a < x < a

ii. $y = \sin^{-1}\left(\frac{x}{\sqrt{a^2+x^2}}\right)$

iii. $y = \cos^{-1}\left(\frac{x}{\sqrt{a^2+x^2}}\right)$

iv. $y = \tan^{-1}\left(\frac{x}{a+\sqrt{a^2-x^2}}\right)$

v. $y = \tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right)$

vi. $y = \tan^{-1}\left(\sqrt{\frac{a-x}{a+x}}\right)$, –a < x < a

ANSWERS

1. **i.** ½ $\tan^{-1} x$ **ii.** $\frac{\pi}{2} - \frac{1}{2}\tan^{-1} x$ **iii.** $\frac{\tan^{-1}\sqrt{x}}{2}$ **iv.** $\frac{\pi}{2} - \frac{\tan^{-1}\sqrt{x}}{2}$ **v.** $\frac{\tan^{-1} x^n}{2}$ **vi.** ½ $\cot^{-1}x$

vii. $\frac{\cot^{-1} x}{2}$ **viii.** $\frac{\pi}{2} - \frac{\cot^{-1} x^n}{2}$ **ix.** $2\tan^{-1} x$ **x.** $2\tan^{-1}(a^x)$ **xi.** $2\tan^{-1} x$

xii. $\frac{\pi}{2} - 2\tan^{-1} x$ **xiii.** $2\tan^{-1}x$ **xiv.** $2\tan^{-1} x^n$ **xv.** $\frac{\pi}{2} - 2\tan^{-1} x$ **xvi.** $3\sin^{-1}x$

xvii. $3\tan^{-1} x$ **xviii.** $\frac{\pi}{2} - 2\sin^{-1}$ **xix.** $\frac{\pi}{4} + \sin^{-1} x$ **xx.** $\frac{\pi}{4} - \sin^{-1} x$ **xxi.** $\sin^{-1}x - \sin^{-1}\sqrt{x}$

xxii. ½ $\cos^{-1} x$ **xxiii.** $\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x$ **xxiv.** $\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$ **xxv.** $\frac{\pi}{4} - \frac{1}{2}\cos^{-1} x$ **xxvi.** $2\sin^{-1}x$

xxvii. $1/4 \cos^{-1} x$ **xxviii.** $\sqrt{1-x^2}$

2. **i.** $\sin^{-1}(x/a)$ **ii.** $\tan^{-1}(x/a)$ **iii.** $\cot^{-1}(x/a)$ **iv.** ½ $\sin^{-1}(x/a)$ **v.** ½ $\tan^{-1}(ax)$ **vi.** ½ $\cos^{-1}(x/a)$

TYPE - 8 PROBLEMS BASED ON SUM AND DIFFERNCE OF TWO SAME INVERSE TRIGONOMETRIC FUNCTIONS

IDENTITIES OF ADDITION AND SUBTRACTION:

A. (i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, $x \ge 0, y \ge 0$ & $(x^2 + y^2) \le 1$

$= \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, $x \ge 0, y \ge 0$ & $x^2 + y^2 \ge 1$

Proof : Let $A = \sin^{-1} x$ and $B = \sin^{-1}y$ where $x, y \in [0, 1]$.

$\sin(A + B) = x\sqrt{1-y^2} + y\sqrt{1-x^2} \Rightarrow \sin^{-1}\sin(A + B) = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$$\Rightarrow \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = \begin{cases} A+B, & \text{for } 0 \le A+B \le \pi/2 \\ \pi-(A+B) & \text{for } \pi/2 \le A+B \le \pi \end{cases} = \begin{cases} \sin^{-1}x+\sin^{-1}y, & x^2+y^2 \le 1 \\ \pi-(\sin^{-1}x+\sin^{-1}y), & x^2+y^2 \ge 1 \end{cases}$$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$; $x, y \in [0, 1]$

(iii) $\cos^{-1} x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$; $x, y \in [0, 1]$

(iv) $$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right); & 0 \le x < y \le 1 \\ -\cos^{-1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\right); & 0 \le y < x \le 1 \end{cases}$$

(v) $$\tan^{-1}x + \tan^{-1}y = \begin{cases} \pi/2 & \text{if } x, y > 0 \text{ \& } xy = 1 \\ -\pi/2 & \text{if } x, y < 0 \text{ \& } xy = 1 \\ \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x, y \ge 0 \text{ \& } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) & \text{if } x, y \ge 0 \text{ \& } xy > 1 \end{cases}$$

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, $x \ge 0, y \ge 0$

1. $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)=\sin^{-1}\left(\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}\right)$

$= \sin^{-1}\left(\frac{3}{5}\cdot\frac{12}{13}+\frac{5}{13}\cdot\frac{4}{5}\right)=\sin^{-1}\left(\frac{56}{55}\right)$ $\left(\because\left(\frac{3}{5}\right)^2+\left(\frac{5}{13}\right)^2\le 1\right)$

2. $\cos^{-1}\left(\frac{3}{5}\right)-\cos^{-1}\left(\frac{8}{17}\right)=-\cos^{-1}\left(\frac{3}{5}\cdot\frac{8}{17}+\sqrt{1-\left(\frac{3}{5}\right)^2}\sqrt{1-\left(\frac{8}{17}\right)^2}\right)$ $\left(\because\frac{3}{5}>\frac{8}{17}\right)$

$= -\cos^{-1}\left(\frac{24}{85}+\frac{\sqrt{25-9}}{5}\cdot\frac{\sqrt{289-64}}{17}\right)$ = $-\cos^{-1}\left(\frac{24}{85}+\frac{4}{5}\cdot\frac{15}{17}\right)=-\cos^{-1}\left(\frac{84}{85}\right)$.

3. Prove that: $\sin^{-1}\left(\frac{3}{5}\right)$ + $\sin^{-1}\left(\frac{8}{17}\right)$ = $\cos^{-1}\left(\frac{84}{85}\right)$.

Sol. L.H.S. = $\sin^{1}\left(\frac{3}{5}\right)$ + $\sin^{-1}\left(\frac{8}{17}\right)$ = $\sin^{-1}\left(\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2}-\frac{8}{17}\sqrt{1-\left(\frac{3}{2}\right)^2}\right)$

$= \sin^{-1}\left(\frac{3}{5}\times\frac{15}{17}-\frac{8}{17}\times\frac{4}{5}\right)=\sin^{-1}\left(\frac{13}{85}\right)=\cos^{-1}\sqrt{1-\left(\frac{13}{85}\right)^2}$ $\left(\because \sin^{-1}x=\cos^{-1}\sqrt{1-x^2}\text{ for }0\le x\le 1\right)$

$= \cos^{-1}\left(\frac{\sqrt{(85)^2-(13)^2}}{85}\right)=\cos^{-1}\left(\frac{\sqrt{7225-169}}{85}\right)=\cos^{-1}\left(\frac{\sqrt{7056}}{85}\right)=\cos^{-1}\left(\frac{84}{85}\right)$

4. $\tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1}\left(\frac{2+3}{1-2\times 3}\right) = \pi + \tan^{-1}(-1) = \pi - \tan^{-1} 1 = \pi - \frac{\pi}{4}=\frac{3\pi}{4}$.

PRACTICE QUESTIONS

1. Prove the following :

i. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$

ii. $\sin^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{8}{17}\right)=\cos^{-1}\left(\frac{36}{85}\right)$

iii. $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$

iv. $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$

v. $\tan^{-1}\left(\frac{2}{11}\right)+\tan^{-1}\left(\frac{7}{24}\right)=\tan^{-1}\left(\frac{1}{2}\right)$

vi. $\tan^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{1}{4}\right)=\frac{\pi}{4}$

vii. $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$

viii. $\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$

ix. $\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$.

x. $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

xi. $\sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\frac{27}{11}$

xii. $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

xiii. $\sin^{-1}\left(\frac{4}{5}\right) + 2\tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$

xiv. $\sin^{-1}\left(\frac{12}{13}\right) + \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right) = \pi$

2. Show that: $\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$.

3. Prove that : $\tan^{-1}\frac{1-x}{1+x}-\tan^{-1}\frac{1-y}{1+y}=\sin^{-1}\frac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}$.

4. Prove that: $\tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{zx}{yr}+\tan^{-1}\frac{xy}{zr}=\frac{\pi}{2}$, if $x^2 + y^2 + z^2 = r^2$.

TYPE - 9 PROBLEMS BASED ON SIMPLYFYING INVERSE TANGENT FUNCTIONS ON FORMULA FOR TAN INVERSE

ILLUSTRATIONS

1. Simplify the expression: $\tan^{-1}\left[\frac{a+bx}{b-ax}\right],\ x < \frac{b}{a}$

Sol. $\tan^{-1}\left[\frac{a+bx}{b-ax}\right],\ x < \frac{b}{a} = \tan^{-1}\left[\frac{\frac{a}{b}+\frac{b}{b}x}{\frac{b}{b}-\frac{a}{b}x}\right] = \tan^{-1}\left[\frac{\frac{a}{b}+x}{1-\frac{a}{b}x}\right]$

$= \tan^{-1}\frac{a}{b} + \tan^{-1}x \quad \{\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\}$

PRACTICE QUESTIONS

1. Simplify the following :

i. $\tan^{-1}\left[\frac{\sqrt{x}+\sqrt{y}}{1-\sqrt{xy}}\right]$

ii. $\tan^{-1}\left[\frac{x+\sqrt{x}}{1-x^{3/2}}\right]$

iii. $y=\tan^{-1}\left(\frac{x^{1/3}+a^{1/3}}{1-(ax)^{1/3}}\right)$

iv. $y=\tan^{-1}\left(\frac{4x}{1-4x^2}\right)$

v. $y=\tan^{-1}\left(\frac{x}{1+6x^2}\right)$

vi. $y=\tan^{-1}\left(\frac{5x}{1-6x^2}\right)$

vii. $y=\tan^{-1}\left(\frac{5ax}{a^2-6x^2}\right)$

viii. $\cot^{-1}(1-x+x^2)+\tan^{-1}x$

ix. $y=\tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$

x. $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right], \frac{-\pi}{2}<x<\frac{\pi}{2}, \frac{a}{b}\tan x>-1$

xi. $y=\tan^{-1}\frac{4x}{1+5x^2}+\tan^{-1}\frac{2+3x}{3-2x}$

ANSWERS

1. **i.** $\tan^{-1}\sqrt{x} - \tan^{-1}\sqrt{y}$ **ii.** $\tan^{-1}x + \tan^{-1}\sqrt{x}$ **iii.** $\tan^{-1}x^{1/3} + \tan^{-1}a^{1/3}$ **iv.** $2\tan^{-1}2x$

v. $\tan^{-1}3x - \tan^{-1}2x$ **vi.** $\tan^{-1}3x + \tan^{-1}2x$ **vii.** $\tan^{-1}\frac{3x}{a} + \tan^{-1}\frac{2x}{a}$ **viii.** $\tan^{-1}(x+1)$

ix. $\tan^{-1}\frac{a}{b} + x$ **x.** $\tan^{-1}\frac{a}{b} - x$ **xi.** $\tan^{-1}5x + \tan^{-1}\frac{2}{3}$

TYPE - 10 PROBLEMS BASED ON SIMPLYFYING MIXED TRIGONOMETRIC FUNCTIONS

ILLUSTRATIONS

1. Prove that: (i) $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) = 7$ (ii) $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right) = \frac{14}{15}$

Sol. (i) L.H.S. = $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) = \cot\left(\frac{\pi}{4} - 2\tan^{-1}\frac{1}{3}\right) = \cot\left\{\frac{\pi}{4} - \tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)\right\}$

$\left(\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} \text{ for } |x|<1\right) = \cot\left(\frac{\pi}{4} - \tan^{-1}\frac{3}{4}\right) = \frac{1}{\tan\left(\frac{\pi}{4} - \tan^{-1}\frac{3}{4}\right)} = \frac{1+\tan\left(\tan^{-1}\frac{3}{4}\right)}{1-\tan\left(\tan^{-1}\frac{3}{4}\right)} = \frac{1+\frac{3}{4}}{1-\frac{3}{4}} = \frac{7}{1} = 7 = \text{R.H.S.}$

(ii) L.H.S. = $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$ = sin (2 θ) + cos φ, where θ = $\tan^{-1}\frac{1}{3}$, θ = $\tan^{-1}2\sqrt{2}$.

$= \frac{2\tan\theta}{1+\tan^2\theta} + \frac{1}{\sec\phi} = \frac{2\tan\theta}{1+\tan^2\theta} + \frac{1}{\sqrt{1+\tan^2\phi}}; \theta\phi \in \left(0, \frac{\pi}{2}\right) = \frac{2\left(\frac{1}{3}\right)}{1+\left(\frac{1}{3}\right)^2} + \frac{1}{\sqrt{1+\left(2\sqrt{2}\right)^2}} = \frac{2}{3}\times\frac{9}{10} + \frac{1}{3} = \frac{3}{5} + \frac{1}{3} = \frac{9+5}{15} = \frac{14}{15}$

2. Prove that: $2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.

Sol. $\sec^{-1}x = \tan^{-1}\sqrt{x^2-1}$ for $x \geq 1$, $\therefore \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) = \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1} = \tan^{-1}\left(\frac{1}{7}\right)$

Hence, L.H.S. = $2\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right) + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right) + \tan^{-1}\frac{1}{7}$

$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{2\times(1/3)}{1-(1/3)^2}\right\} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{21+4}{28-3}\right) = \tan^{-1}1 = \frac{\pi}{4} = \text{R.H.S.}$

PRACTICE QUESTIONS

1. Prove that :

i. $\cos^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right) = \frac{4}{5}$

ii. $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right) = \frac{17}{6}$

iii. $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$.

iv. $2\tan^{-1}\sqrt{\frac{b}{a}} = \cos^{-1}\frac{a-b}{a+b}$

v. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left[\frac{1-x}{1+x}\right]$

vi. $\tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cot^{-1}\left[\frac{1-x^2}{2x}\right] = \frac{\pi}{2}$

vii. $\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right] = 1$

viii. $\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right) = 0$

ix. $\cot\left\{\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)\right\}+\cos^{-1}(1-2x^2)+\cos^{-1}(2x^2-1)=\pi$

x. $\sin\left[\cot^{-1}\{\cos(\tan^{-1}x)\}\right] = \sqrt{\frac{x^2+1}{x^2+2}}$

xi. $\cos\left[\tan^{-1}\{\sin(\cot^{-1}x)\}\right] = \sqrt{\frac{x^2+1}{x^2+2}}$

TYPE - 11 PROBLEMS BASED ON SOLVING INVERSE TRIGONOMETRIC FUNCTIONS.

ILLUSTRATIONS

1. Find x if:

(i) $\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$ (ii) $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{2}$ (iii) $\sin^{-1}\left(\cos^{-1}\frac{1}{5}+\sin^{-1}x\right)=1$

Sol. (i) Given $\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}\Rightarrow\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}-\sin^{-1}\left(\frac{5}{x}\right)$

$\Rightarrow\frac{12}{x}=\sin\left(\frac{\pi}{2}=\sin^{-1}\left(\frac{5}{x}\right)\right)\Rightarrow\frac{12}{x}=\cos\left(\sin^{-1}\left(\frac{5}{x}\right)\right)=\sqrt{1-\left(\frac{5}{x}\right)^2}$ $\left(\because\cos(\sin^{-1}x)=\sqrt{1-x^2},|x|\le 1\right)$

$\Rightarrow\left(\frac{12}{x}\right)^2=1-\left(\frac{5}{x}\right)^2$ $\Rightarrow x^2 = 144 + 25 = 169$ $\Rightarrow x = 13$.

($\because$ x = –13 does not satisfy the given equation)

(ii) Given $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$ $\Rightarrow \sin^{-1}2x = \frac{\pi}{3} - \sin^{-1}x$

$\Rightarrow 2x = \sin\left(\frac{\pi}{3}-\sin^{-1}x\right)$ $\Rightarrow 2x = \sin\frac{\pi}{3}\cos(\sin^{-1}x) - \cos\frac{\pi}{3}\sin(\sin^{-1}x)$

$\Rightarrow 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2}-\frac{1}{2}x$ $\Rightarrow 2x+\frac{x}{2}=\frac{\sqrt{3}}{2}\sqrt{1-x^2}$.

On squaring the two sides, we get, $\frac{25x^2}{4}=\frac{3}{4}(1-x^2)\Rightarrow 28x^2 = 3 \Rightarrow x^2 = \frac{3}{28}$ $\Rightarrow x = \sqrt{\frac{3}{28}}$

$\left(\because x = -\sqrt{\frac{3}{28}}\text{ does not satisfy the given equation}\right)$

(iii) Given $\sin\left(\cos^{-1}\frac{1}{5}+\sin^{-1}x\right)=1 \Rightarrow \cos^{-1}\frac{1}{5}+\sin^{-1}x=\frac{\pi}{2}$ $\left(0\le\cos^{-1}\frac{1}{5}\le\frac{\pi}{2}\text{ and }-\frac{\pi}{2},\ \therefore -\frac{\pi}{2}\le\cos^{-1}\frac{1}{5}+\sin^{-1}x\le\pi\right)$

PRACTICE QUESTIONS

1. Solve for x:

i. $\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$ **ii.** $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$

iii. $\tan^{-1}\frac{x-1}{x+1}+\tan^{-1}\frac{2x-1}{2x+1}=\tan^{-1}\frac{23}{36}$ **iv.** $\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}$, where x > 0.

v. $\sin\{\sin^{-1}1/5+\cos^{-1}x\}=1$ **vi.** $\sin^{-1}x-\cos^{-1}x=\frac{\pi}{6}$ **vii.** $\cos^{-1}5/14=\tan^{-1}x$

viii. $\tan^{-1}4+\cot^{-1}x=\frac{\pi}{2}$. **ix.** $\cos^{-1}\left(\frac{1-y}{1+y}\right)=2\tan^{-1}x$ **x.** $\sin^{-1}\frac{3x}{5}+\sin^{-1}\frac{4x}{5}=\sin^{-1}x$

xi. $\sin^{-1}6x+\sin^{-1}6\sqrt{3}\,x=-\frac{\pi}{2}$ **xii.** $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

xiii. $2\tan^{-1}(\cos x)=\tan^{-1}(2\,\mathrm{cosec}\,x)$ **xiv.** $2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x)$, $0<x<\frac{\pi}{2}$.

xv. $\tan(\cos^{-1}x)=\sin\left(\cot^{-1}\frac{1}{2}\right)$ **xvi.** $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}]=0$

xvii. $3\sin^{-1}\frac{2x}{1+x^2}-4\cos^{-1}\frac{1-x^2}{1+x^2}+2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$ **xviii.** $\tan^{-1}\sqrt{x^2+x}\ \sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

2. Solve the equation: $\cos^{-1}\left(\frac{a}{x}\right)-\cos^{-1}\left(\frac{b}{x}\right)=\cos^{-1}\left(\frac{1}{b}\right)-\cos^{-1}\left(\frac{1}{a}\right),|a|\ge 1,|b|\ge 1$.

ANSWERS

1. **i.** $x=\pm\frac{1}{\sqrt{2}}$ **ii.** x = 1/6 **iii.** $x=\frac{4}{3}$ **iv.** $x=\sqrt{3}$ **v.** $x=\frac{1}{5}$ **vi.** $x=\frac{\sqrt{3}}{2}$ **vii.** $x=\frac{12}{5}$

viii. x = 4 **ix.** $x=\sqrt{y}$ **x.** x = –1, 0, 1 **xi.** x = –1/12 **xii.** x = 0 **xiii.** $x=\frac{\pi}{4}$

xiv. $x=\pi/4$ **xv.** $x=\pm\frac{\sqrt{5}}{3}$ **xvi.** $x=\pm 1,\ -1\pm\sqrt{2},\ 1\pm\sqrt{2}$ **xvii.** $x=1/\sqrt{3}$ **xviii.** x = 0, –1

2. $x=\pm ab$.

MIX PROBLEMS

ILLUSTRATIONS

1. If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, then find x.

Sol. We have $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$. As $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$, so $\cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x$

$\therefore (\tan^{-1}x)^2+\left(\frac{\pi}{2}-\tan^{-1}x\right)=\frac{5\pi^2}{8}\ \Rightarrow (\tan–1\ x)^2+\left(\frac{\pi}{2}\right)^2-2\times\frac{\pi}{2}\tan^{-1}x+(\tan^{-1}x)^2=\frac{5\pi^2}{8}$

$\Rightarrow 2(\tan^{-1}x)^2-\pi\tan^{-1}x=\frac{5\pi^2}{8}-\frac{\pi^2}{4}\Rightarrow 2(\tan^{-1}x)^2-\pi\tan^{-1}x=\frac{3\pi^2}{8}\Rightarrow 16(\tan^{-1}x)^2-8\pi\tan^{-1}x-3\pi^2=0$

$\therefore \tan^{-1}x=\frac{8\pi\pm\sqrt{(-8\pi_{-}^{2}-4(16)(-3\pi^2)}}{2(16)}$

$\Rightarrow \tan^{-1} x = \dfrac{8\pi \pm \sqrt{64\pi^2 + 192\pi^2}}{32} = \dfrac{8\pi \pm \sqrt{256\pi^2}}{32}$ $\Rightarrow \tan^{-1} x = \dfrac{8\pi \pm 16\pi}{32} = \dfrac{24\pi}{32}, \dfrac{-8\pi}{32} = \dfrac{3\pi}{4}, -\dfrac{\pi}{4}$

$\therefore$ x = – tan $\dfrac{\pi}{4}$, – tan $\dfrac{\pi}{4}$ [$\because$ tan(π – θ) = – tan θ] $\Rightarrow$ x = –1, Hence x = –1.

2. Prove that $\tan\left(\dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right) + \tan\left(\dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right) = \dfrac{2b}{a}$.

Sol. Put $\dfrac{1}{2}\cos^{-1}\dfrac{a}{b} = \theta$ .Given expression = $\tan\left(\dfrac{\pi}{4} + \theta\right) + \tan\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{1+\tan\theta}{1-\tan\theta} + \dfrac{1-\tan\theta}{1+\tan\theta}$

$= \dfrac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)} = \dfrac{2 + 2\tan^2\theta}{1-\tan^2\theta} = 2\left(\dfrac{1+\tan^2\theta}{1-\tan^2\theta}\right) = \dfrac{2}{\cos 2\theta} = \dfrac{2}{\frac{a}{b}}$

$\left[\because \dfrac{1}{2}\cos^{-1}\dfrac{a}{b} = \theta \Rightarrow \cos^{-1}\dfrac{a}{b} = 2\theta \Rightarrow \cos 2\theta = \dfrac{a}{b}\right] = \dfrac{2b}{a}$ = RHS.

3. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

Sol. Let $\sin^{-1} x = \alpha$, $\sin^{-1} y = \beta$ and $\sin^{-1} z = \gamma$ $\therefore$ x = sin α, y = sin β and z = sin γ

LHS = $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$ = sin α cos α + sin β cos β + sin γ cos γ

$= \frac{1}{2}$(2 sinα cos α +2 sin β sin β +2sin γ cos γ) $= \frac{1}{2}$(sin 2α sin 2β + sin 2γ) $\frac{1}{2}$[(sin 2α + sin 2β) + sin 2γ)

$= \dfrac{1}{2}\left[2\sin\dfrac{2\alpha+2\beta}{2}\cos\dfrac{2\alpha-2\beta}{2} + 2\sin\gamma\cos\gamma\right] = \dfrac{1}{2}$[2sin (α + β) cos (α – β) 2γ) + sin γ cos γ)

$= \dfrac{1}{2}$[2sin γ cos (α – β) + 2 sin γ cos γ) [$\because$ α + β + γ = π]

= sin γ [cos (α–β) + cos γ]

= sin γ [cos (α – β) – cos (π + γ)] [– cos θ = cos (π – θ)]

= sin γ [cos (α – β) – cos (α + β)] [$\because$ α + β + γ = π]

= sin γ [2 sin α sin β] = 2 sin α sin β sin γ = 2xyz = RHS. Hence, proved.

MISCELLANEOUS QUESTIONS

1. Prove : $\cot^{-1}\left[2\tan\left(\cos^{-1}\dfrac{8}{17}\right)\right] + \tan^{-1}\left[2\tan\left(\sin^{-1}\dfrac{8}{17}\right)\right] = \tan^{-1}\left(\dfrac{300}{161}\right)$

2. If $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right] = \beta$ then prove that $x^2 = \sin^2\beta$.

3. Prove : $\tan^{-1}\left\{\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\} = \dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}x,\ 0 < x < 1$

4. Prove : $\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\} = \dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}x^2 \quad -1 < x < 1$

5. If $\sin^{-1}\dfrac{2a}{1+a^2} - \cos^{-1}\dfrac{1-b^2}{1+b^2} = \tan^{-1}\dfrac{2x}{1-x^2}$, then prove that $x = \dfrac{a-b}{1+ab}$.

6. If $\sin^{-1}\dfrac{2a}{1+a^2} + \sin^{-1}\dfrac{2b}{1+b^2} = 2\tan^{-1}x$, prove that $x = \dfrac{a+b}{1-ab}$.

7. Prove that : $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]=\frac{x+y}{1-xy}$, if | x | < 1, y > 0 and xy < 1.

8. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, then prove that : $x+y+z=xyz$

9. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

10. If $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$, prove that $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} = \sin^2\alpha$.

11. Prove the follwing: :

i. $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$ **ii.** $\cos^{-1}x+\cos^{-1}y=\cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2})$

iii. $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

12. Prove that : $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}+\tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\}=\frac{2b}{a}$

13. If $y = \cot^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right)$ prove that : $\sin y = \tan^2\frac{x}{2}$

14. Prove : $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) - \cot^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right) = \frac{\pi}{4}$ $x = (0, \pi/2)$

15. Solve $\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) - \cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = 2\tan^{-1}x$.

16. Prove that: $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\ldots+\tan^{-1}\left(\frac{1}{n^2+n+1}\right)+\ldots\infty=\frac{\pi}{4}$.

17. Prove that: (i) $\tan^{-1}n + \cot^{-1}(n + 1) = \tan^{-1}(n^2 + n + 1)$ (ii) $\tan^{-1}\left(\frac{n}{n+1}\right)-\tan^{-1}(2n+1)=\frac{3\pi}{4}$

18. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$.

19. Find the greatest and least values of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$.

20. Find the maximum and minimum values of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$, where $-1 \le x \le 1$.

21. If a, b, c > 0 such that a + b + c = abc, find the value of $\tan^{-1}a + \tan^{-1}b + \tan^{-1}c$.

22. If a, b, c > 0 and $s=\frac{a+b+c}{2}$, prove that $\tan^{-1}\sqrt{\frac{2as}{bc}}+\tan^{-1}\sqrt{\frac{2bs}{ca}} = \tan^{-1}\sqrt{\frac{2cs}{ab}} = \pi$.

23. Evaluate : $\tan^{-1}\left(\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$, where $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$.

24. Prove that : $\cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right) = 2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$.

25. Show that : $2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\} = \tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right)$.

26. Prove that : $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^2}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{2}\right) = (\alpha+\beta)\left(\alpha^2+\beta^2\right)$.

27. Prove that : $\tan^{-1}\left\{\frac{\cos 2\alpha\sec 2\beta+\cos 2\beta\sec 2\alpha}{2}\right\} = \tan^{-1}\left\{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)\right\}+\tan^{-1}1$

NCERT EXEMPLAR QUESTIONS

1. Find the value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

2. Evaluate $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$.

3. Prove that $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7$.

4. Find the value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$.

5. Find the value of $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$.

6. Show that $2\tan^{-1}(-3) = -\frac{\pi}{2} + \tan^{-1}\left(\frac{-4}{3}\right)$.

7. Find the real solution of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$.

8. Find the value of $\sin\left(2\tan^{-1}\frac{1}{3}\right)+\cos\left(\tan^{-1}2\sqrt{2}\right)$.

9. If $2\tan^{-1}(\cos\theta) = \tan^{-1}(2\operatorname{cosec}\theta)$, then show that $\theta=\frac{\pi}{4}$.

10. Show that $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$.

11. Solve the equation $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$.

12. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$.

13. Find the simplified form of $\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$, where $x\in\left[\frac{-3\pi}{4},\frac{\pi}{4}\right]$.

14. Prove that $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{77}{85}$.

15. Show that $\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$.

16. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\sin^{-1}\frac{1}{\sqrt{5}}$.

17. Find the value of $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$.

18. Show that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$ and justify why the other value $\frac{4+\sqrt{7}}{3}$ is ignored?

19. If a_1, a_2, a_3, ..., a_n is an A.P. with common difference d, then evaluate the following expression.

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+...+\tan^{-1}\left(\frac{d}{1+a_{n-1}\,a_n}\right)\right]$$

20. Which of the following is the principal value branch of $\cos^{-1} x$?

a. $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ **b.** $(0, \pi)$ **c.** $[0, \pi]$ **d.** $(0,\pi)-\frac{\pi}{2}$

21. Which of the following is the principal value branch of $\text{cosec}^{-1} x$?

a. $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ **b.** $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ **c.** $\left[\frac{\pi}{2},\frac{\pi}{2}\right]$ **d.** $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]-[0]$

22. If $3\tan^{-1} x + \cot^{-1} x = \pi$, then x equal to

a. 0 **b.** 1 **c.** –1 **d.** 1/2

23. The value of $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ is:

a. $\frac{3\pi}{5}$ **b.** $\frac{-7\pi}{5}$ **c.** $\frac{\pi}{10}$ **d.** $\frac{-\pi}{10}$

24. The domain of the function $\cos^{-1}(2x-1)$ is:

a. [0, 1] **b.** [–1, 1] **c.** (–1, 1) **d.** [0, π]

25. The domain of the function defined by f(x) = $\sin^{-1}\sqrt{x-1}$ is:

a. [1, 2] **b.** [–1, 1] **c.** [0, 1] **d.** None of these

26. If $\cos\left(\sin^{-1}\frac{2}{5}+\cos^{-1}x\right)=0$, then x is equal to:

a. 1/5 **b.** 2/5 **c.** 0 **d.** 1

27. The value of $\sin[2\tan^{-1}(0.75)]$ is:

a. 0.75 **b.** 1.5 **c.** 0.96 **d.** sin 1.5

28. The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is:

a. $\frac{\pi}{2}$ **b.** $\frac{3\pi}{2}$ **c.** $\frac{5\pi}{2}$ **d.** $\frac{7\pi}{2}$

29. The value of 2 $\sec^{-1} 2 + \sin^{-1}\left(\frac{1}{2}\right)$ is:

a. $\frac{\pi}{6}$ **b.** $\frac{5\pi}{6}$ **c.** $\frac{7\pi}{6}$ **d.** 1

30. If $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$, then $\cot^{-1} x + \cot^{-1} y$ equals to:

a. $\frac{\pi}{5}$ **b.** $\frac{2\pi}{5}$ **c.** $\frac{3\pi}{5}$ **d.** π

31. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where a, x $\in$] 0, 1[, then the value of x is:

a. 0 **b.** $\frac{a}{2}$ **c.** a **d.** $\frac{2a}{1-a^2}$

32. The value of $\cot\left[\cos^{-1}\left(\frac{7}{25}\right)\right]$ is:

a. $\frac{25}{24}$ **b.** $\frac{25}{7}$ **c.** $\frac{24}{25}$ **d.** $\frac{7}{24}$

33. The value of $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ is:

a. $2+\sqrt{5}$ **b.** $\sqrt{5}-2$ **c.** $\frac{\sqrt{5}+2}{2}$ **d.** $5+\sqrt{2}$

34. If $|x| \le 1$, then $2\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

a. $4\tan^{-1} x$ **b.** 0 **c.** $\frac{\pi}{2}$ **d.** π

35. If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, then $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ equals to:

a. 0 **b.** 1 **c.** 6 **d.** 12

36. The number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2}, \pi\right]$ is:

a. 0 **b.** 1 **c.** 2 **d.** ∞

37. If $\cos^{-1} x > \sin^{-1} x$, then:

a. $\frac{1}{\sqrt{2}} < x \le 1$ **b.** $0 \le x < \frac{1}{\sqrt{2}}$ **c.** $-1 \le x < \frac{1}{\sqrt{2}}$ **d.** $x > 0$

38. The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is ....

39. The value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is ....

40. If $\cos\left(\tan^{-1} x + \cot^{-1}\sqrt{3}\right) = 0$, then the value of x is ....

41. The set of values of $\sec^{-1}\frac{1}{2}$ is ....

42. The principal value of $\tan^{-1}\sqrt{3}$ is ....

43. The value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is ....

44. The value of $\cos(\sin^{-1} x + \cos^{-1} x)$, where $|x| \le 1$, is ....

45. The value of $\tan\left(\frac{\sin^{-1} x + \cos^{-1} x}{2}\right)$, when $x = \frac{\sqrt{3}}{2}$, is ....

46. If $y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then .... < y < ....

47. The result $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when the value of xy is ....

48. The value of $\cot^{-1}(-x)$ $x \in R$ in terms of $\cot^{-1} x$ is ....

49. All trigonometric functions have inverse over their respective domains.

50. The value of the expression $(\cos^{-1} x)^2$ is equal to $\sec^2 x$.

51. The domain of trigonomitric functions can be restricted to any one of their branch (not necesarily principal value) in order to obtain their inverse functions.

52. The least numerical value, either positive or negative of angle θ is called principal value of the inverse trigonometric function.

53. The graph of inverse trigonometric function can be obtained from the graph of their corresponding function by interchanging X and Y-axes.

54. The minimum value of n for which $\tan^{-1}\frac{n}{x} > \frac{\pi}{4}, n \in N$, is valid is 5.

55. The principal value of $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ is $\frac{\pi}{3}$.

PREVIOUS YEARS BOARD QUESTIONS

1. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x. **[Delhi 2014]**

2. If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$; $xy < 1$, then write the value of $x + y + xy$. **[A.I. 2014]**

3. Write the value of $\cos^{-1}\left(-\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$. **[Foreign 2014]**

4. Write the principal value of $\cos^{-1}[\cos(680)^{o}]$. **[Delhi 2014C]**

5. Write the principal value of $\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$. **[A.I 2014C]**

6. Find the value of the following: $\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$. **[A.I 2014C]**

7. Write the principal value of $\left[\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\left(-\frac{1}{2}\right)\right]$. **[Delhi 2013C]**

8. Write the valueof $\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right)$. **[Delhi 2013C]**

9. Write the principal value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$. **[Hots; Delhi 2013]**

10. Write the value of $\tan\left(2\tan^{-1}\frac{1}{2}\right)$. **[Delhi 2013]**

11. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$. **[A.I. 2013]**

12. Write the principal value of $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)$. **[A.I. 2013]**

13. Write the value of $\cos^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(-\frac{1}{2}\right)$. **[Delhi 2012]**

14. Find the principal value of $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$. **[A.I. 2012]**

15. Using the principal values, write the value of $\cos^{-1}\left(\frac{1}{2}\right)+2\sin^{-1}\left(\frac{1}{2}\right)$. **[A.I. 2012C]**

16. Write the value of $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$. **[Delhi 2011]**

17. Write the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$. **[Delhi 2011]**

18. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$. **[Delhi 2011, 2009]**

19. What is the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{6}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$? **[A.I. 2011, 08, 09C]**

20. What is the principal value of $\tan^{-1}(-1)$? **[Foreign 2011, 2008C]**

21. Using the principal values, write the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. **[Hots, A.I. 2011C]**

22. Write the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$. **[Delhi 2011C]**

23. Write the principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$. **[Delhi 2010]**

24. What is the principal value of $\sec^{-1}(-2)$? **[A.I. 2010]**

25. What is the domain of the function $\sin^{-1} x$? **[Foreign 2010]**

26. Using the principal values, find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$. **[A.I. 2010C]**

27. If $\tan^{-1}\left(\sqrt{3}\right) + \cot^{-1} x = \frac{\pi}{2}$, then find the value of x. **[A.I. 2010C]**

28. Write the principal value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$. **[Delhi 2009]**

29. Using the principal values, evaluate $\tan^{-1}(1) + \sin^{-1}\left(-\frac{1}{2}\right)$. **[Delhi 2009C]**

30. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$. **[A.I. 2008C]**

31. Solve the following equation: $\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$. **[Foreign 2014; A.I. 2013]**

32. Solve following equation for x, $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1} x$. **[A.I. 2014C Foreign 2011C]**

33. Solve for x, $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$. **[Delhi 2014C]**

34. Solve for x, $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$. **[A.I. 2014]**

35. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2};\ x\in\left(0,\frac{\pi}{4}\right)$. **[Delhi 2014, 2011, A.I. 2009]**

36. Prove that $2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$. **[Delhi 2014]**

37. Prove that $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, \frac{-1}{\sqrt{2}}\le x\le 1$. **[A.I. 2014, 2011]**

38. If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, then find the value of x. **[A.I 2014]**

39. Prove that $\cos^{-1}(x)+\cos^{-1}\left\{\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right\}=\frac{\pi}{3}$. **[A.I. 2014C]**

40. Prove that $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18=\cot^{-1}3$. **[Foreign 2014]**

41. Prove the following: $\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]=\frac{x}{2}; x\in\left(0,\frac{\pi}{4}\right)$. **[Delhi 2014, 2011]**

42. Prove that, $\sin^{-1}\left(\frac{8}{17}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\cos^{-1}\left(\frac{36}{85}\right)$. **[A.I. 2014C]**

43. Show that, $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)=\frac{4-\sqrt{7}}{3}$. **[A.I. 2013]**

44. Prove that, $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$. **[Delhi 2013C]**

45. Solve for x, $\tan^{-1}3x+\tan^{-1}2x=\frac{\pi}{4}$. **[Delhi 2013C]**

46. Find the value of the following: $\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$, if $|x|<1$, $y>0$ & $xy<1$. **[Delhi 2013]**

47. Prove that: $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$. **[Delhi 2013, A.I. 2011]**

48. Prove that: $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)=\frac{\pi}{4}-\frac{x}{2}, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. **[Delhi 2012]**

49. Prove that: $\cos^{-1}\left(\frac{4}{5}\right)\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$. **[A.I. 2012]**

50. Prove that $\sin^{-1}\left(\frac{63}{65}\right)=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)$. **[Foreign 2012]**

51. Solve for x, $2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x), x\neq\frac{\pi}{2}$. **[Foreign 2012]**

52. Find the value of $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$. **[Delhi 2011]**

53. Prove that $2\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\tan^{-1}\left(\frac{31}{17}\right)$. **[A.I. 2011]**

54. Prove that $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$. **[Foreign 2011]**

55. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\tan^{-1}\frac{4}{3}$. **[A.I. 2011C]**

56. Solve for x, cos(2 $\sin^{-1}$ x) = $\frac{1}{9}$, x > 0. **[A.I. 2011C]**

57. Prove that $2\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{17}{13}=\frac{\pi}{4}$. **[Delhi 2011C]**

58. Solve for x, $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$. **[Delhi 2011C]**

59. Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in(0,1)$. **[Delhi 2010]**

60. Prove that $\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$. **[Delhi 2010]**

61. Prove that $\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$. **[A.I. 2010]**

62. Prove that cos[$\tan^{-1}$ {sin($\cot^{-1}$ x)}] = $\sqrt{\frac{1+x^2}{2+x^2}}$. **[A.I. 2010]**

63. Solve for x, $\cos^{-1}$ x + $\sin^{-1}\left(\frac{x}{2}\right)=\frac{\pi}{6}$. **[A.I. 2010C]**

64. Prove that $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$. **[A.I. 2010C]**

65. Solve for x, $\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4}, 0<x<\sqrt{6}$. **[Delhi 2010C]**

66. Solve for x, $\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right)$ x > 0. **[Delhi 2010C]**

67. Prove that $\sin^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{16}{65}\right)=\frac{\pi}{2}$. **[Delhi 2009]**

68. Prove that $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{8}{19}\right)=\frac{\pi}{4}$. **[A.I. 2009C]**

69. Prove that $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{8}\right)\frac{\pi}{4}$. **[A.I. 2009C, Delhi 2008]**

70. Solve for x, $\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}$ + $\tan^{-1}$ x, 0 < x < 1. **[Delhi 2008C]**

71. Prove that : $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$. **[Delhi 2015]**

NCERT EXEMPLAR QUESTIONS ANSWERS

1. 0 **2.** –1 **4.** $-\frac{\pi}{12}$ **5.** $-\frac{\pi}{3}$ **7.** x = 0, –1. **8.** $\frac{14}{15}$ **11.** $-\frac{3}{4}, \frac{3}{4}$

13. $\tan^{-1}\frac{3}{5} - x$ **17.** $\frac{\pi}{4}$ **19.** $\frac{a_n - a_1}{1 + a_n \cdot a_1}$ **20.** c **21.** d

22. b **23.** d **24.** a **25.** a **26.** b **27.** c **28.** a

29. b **30.** a **31.** d **32.** d **33.** b **34.** a **35.** c

36. a **37.** c **38.** $\frac{2\pi}{3}$ **39.** $\frac{2\pi}{5}$ **40.** $\sqrt{3}$ **41.** R – (–1, 1)

42. $\left(\frac{\pi}{3}\right)$ **43.** $\frac{2\pi}{3}$ **44.** 0 **45.** 1 **46.** $-2\pi < y < 2\pi$ **47.** x y > – 1

48. $\pi - \cot^{-1} x, x \in R$ **49.** False **50.** False **51.** True **52.** True **53.** True

54. False **55.** True

PREVIOUS YEARS BOARD QUESTIONS ANSWERS

1. $x = \frac{1}{5}$ **2.** 1 **3.** π **4.** 40º **5.** $\frac{-\pi}{4}$ **6.** $\sqrt{3}$ **7.** $\frac{5\pi}{6}$ **8.** $\frac{\pi}{4}$ **9.** $\frac{11\pi}{12}$

10. $\frac{5}{12}$ **11.** $\frac{\pi}{3}$ **12.** $\frac{\pi}{2}$ **13.** $\frac{2\pi}{3}$ **14.** $\frac{-\pi}{3}$ **15.** $\frac{2\pi}{3}$ **16.** 1 **17.** $\frac{\pi}{4}$ **18.** $\frac{5\pi}{6}$

19. π **20.** $-\frac{\pi}{4}$ **21.** $-\frac{\pi}{3}$ **22.** $-\frac{\pi}{6}$ **23.** $\frac{\pi}{3}$ **24.** $\frac{2\pi}{3}$ **25.** $-1 \le x \le 1$

26. $\frac{\pi}{6}$ **27.** $x = \sqrt{3}$ **28.** $\frac{2\pi}{5}$ **29.** $\frac{\pi}{12}$ **30.** $\frac{\pi}{6}$ **31.** $x = \frac{3}{4}$ **32.** $x = \frac{1}{\sqrt{3}}$ **33.** $x = \frac{\pi}{4}$

34. $x = \sqrt{3}$ **38.** $x = \pm\sqrt{3}$ **45.** $\frac{1}{6}$ **46.** $\frac{x+y}{1-xy}$ **51.** $\frac{\pi}{4}$ **52.** $\frac{\pi}{4}$ **56.** $x = \frac{2}{3}$

58. $x = 2 - \sqrt{3}$ **63.** x = 1 **65.** x = 1 **66.** x = 1/4 **70.** x = $\pm$ 1/2