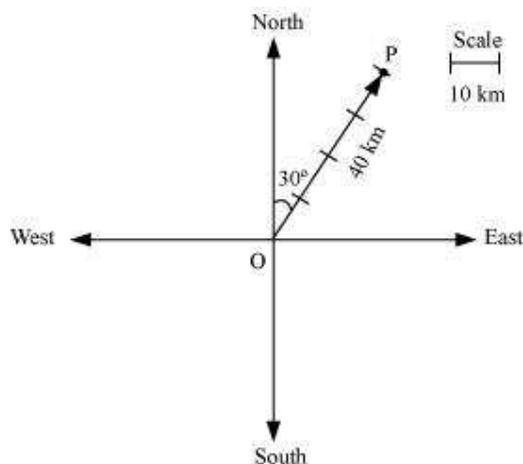


Exercise 10.1**Question 1:**

Represent graphically a displacement of 40 km, 30° east of north.

Answer



Here, vector \overline{OP} represents the displacement of 40 km, 30° East of North.

Question 2:

Classify the following measures as scalars and vectors.

- (i) 10 kg (ii) 2 metres north-west (iii) 40°
(iv) 40 watt (v) 10^{-19} coulomb (vi) 20 m/s^2

Answer

- (i) 10 kg is a scalar quantity because it involves only magnitude.
(ii) 2 meters north-west is a vector quantity as it involves both magnitude and direction.
(iii) 40° is a scalar quantity as it involves only magnitude.
(iv) 40 watts is a scalar quantity as it involves only magnitude.
(v) 10^{-19} coulomb is a scalar quantity as it involves only magnitude.
(vi) 20 m/s^2 is a vector quantity as it involves magnitude as well as direction.

Question 3:

Classify the following as scalar and vector quantities.

- (i) time period (ii) distance (iii) force

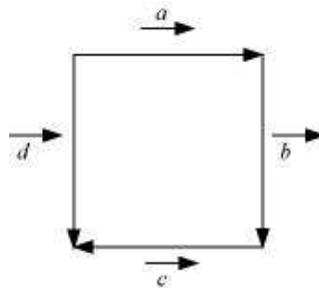
(iv) velocity (v) work done

Answer

- (i) Time period is a scalar quantity as it involves only magnitude.
- (ii) Distance is a scalar quantity as it involves only magnitude.
- (iii) Force is a vector quantity as it involves both magnitude and direction.
- (iv) Velocity is a vector quantity as it involves both magnitude as well as direction.
- (v) Work done is a scalar quantity as it involves only magnitude.

Question 4:

In Figure, identify the following vectors.



- (i) Coinitial (ii) Equal (iii) Collinear but not equal

Answer

- (i) Vectors \vec{a} and \vec{d} are coinitial because they have the same initial point.
- (ii) Vectors \vec{b} and \vec{d} are equal because they have the same magnitude and direction.
- (iii) Vectors \vec{a} and \vec{c} are collinear but not equal. This is because although they are parallel, their directions are not the same.

Question 5:

Answer the following as true or false.

- (i) \vec{a} and $-\vec{a}$ are collinear.
- (ii) Two collinear vectors are always equal in magnitude.
- (iii) Two vectors having same magnitude are collinear.
- (iv) Two collinear vectors having the same magnitude are equal.

Answer

- (i) True.

Vectors \vec{a} and $-\vec{a}$ are parallel to the same line.

(ii) False.

Collinear vectors are those vectors that are parallel to the same line.

(iii) False.

Exercise 10.2

Question 1:

Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \quad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

Answer

The given vectors are:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}; \quad \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}; \quad \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$\begin{aligned} |\vec{b}| &= \sqrt{(2)^2 + (-7)^2 + (-3)^2} \\ &= \sqrt{4 + 49 + 9} \\ &= \sqrt{62} \end{aligned}$$

$$\begin{aligned} |\vec{c}| &= \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2} \\ &= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1 \end{aligned}$$

Question 2:

Write two different vectors having same magnitude.

Answer

$$\text{Consider } \vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k}) \text{ and } \vec{b} = (2\hat{i} + \hat{j} - 3\hat{k}).$$

$$\text{It can be observed that } |\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14} \text{ and}$$

$$|\vec{b}| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}.$$

Hence, \vec{a} and \vec{b} are two different vectors having the same magnitude. The vectors are different because they have different directions.

Question 3:

Write two different vectors having same direction.

Answer

Consider $\vec{p} = (\hat{i} + \hat{j} + \hat{k})$ and $\vec{q} = (2\hat{i} + 2\hat{j} + 2\hat{k})$.

The direction cosines of \vec{p} are given by,

$$l = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}, \quad m = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}, \quad \text{and } n = \frac{1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}.$$

The direction cosines of \vec{q} are given by

$$l = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}, \quad m = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}},$$

$$\text{and } n = \frac{2}{\sqrt{2^2 + 2^2 + 2^2}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

The direction cosines of \vec{p} and \vec{q} are the same. Hence, the two vectors have the same direction.

Question 4:

Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal

Answer

The two vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ will be equal if their corresponding components are equal.

Hence, the required values of x and y are 2 and 3 respectively.

Question 5:

Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (-5, 7).

Answer

The vector with the initial point P (2, 1) and terminal point Q (-5, 7) can be given by,

$$\vec{PQ} = (-5 - 2)\hat{i} + (7 - 1)\hat{j}$$

$$\Rightarrow \vec{PQ} = -7\hat{i} + 6\hat{j}$$

Hence, the required scalar components are -7 and 6 while the vector components are $-7\hat{i}$ and $6\hat{j}$.

Question 6:

Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

Answer

The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

$$\begin{aligned}\therefore \vec{a} + \vec{b} + \vec{c} &= (1-2+1)\hat{i} + (-2+4-6)\hat{j} + (1+5-7)\hat{k} \\ &= 0\hat{i} - 4\hat{j} - 1\hat{k} \\ &= -4\hat{j} - \hat{k}\end{aligned}$$

Question 7:

Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Answer

The unit vector \hat{a} in the direction of vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ is given by $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$$

Question 8:

Find the unit vector in the direction of vector \overline{PQ} , where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.

Answer

The given points are P (1, 2, 3) and Q (4, 5, 6).

$$\therefore \overline{PQ} = (4-1)\hat{i} + (5-2)\hat{j} + (6-3)\hat{k} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$|\overline{PQ}| = \sqrt{3^2 + 3^2 + 3^2} = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$$

Hence, the unit vector in the direction of \overline{PQ} is

$$\frac{\overline{PQ}}{|\overline{PQ}|} = \frac{3\hat{i} + 3\hat{j} + 3\hat{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Question 9:

For given vectors, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$, find the unit vector in the direction of the vector $\vec{a} + \vec{b}$

Answer

The given vectors are $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} - \hat{k}$.

$$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{b} = -\hat{i} + \hat{j} - \hat{k}$$

$$\therefore \vec{a} + \vec{b} = (2-1)\hat{i} + (-1+1)\hat{j} + (2-1)\hat{k} = 1\hat{i} + 0\hat{j} + 1\hat{k} = \hat{i} + \hat{k}$$

$$|\vec{a} + \vec{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Hence, the unit vector in the direction of $(\vec{a} + \vec{b})$ is

$$\frac{(\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|} = \frac{\hat{i} + \hat{k}}{\sqrt{2}} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

Question 10:

Find a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units.

Answer

$$\text{Let } \vec{a} = 5\hat{i} - \hat{j} + 2\hat{k}.$$

$$\therefore |\vec{a}| = \sqrt{5^2 + (-1)^2 + 2^2} = \sqrt{25 + 1 + 4} = \sqrt{30}$$

$$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}}$$

Hence, the vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude 8 units is given by,

$$8\hat{a} = 8 \left(\frac{5\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{30}} \right) = \frac{40}{\sqrt{30}}\hat{i} - \frac{8}{\sqrt{30}}\hat{j} + \frac{16}{\sqrt{30}}\hat{k}$$

$$= 8 \left(\frac{5\vec{i} - \vec{j} + 2\vec{k}}{\sqrt{30}} \right)$$

$$= \frac{40}{\sqrt{30}}\vec{i} - \frac{8}{\sqrt{30}}\vec{j} + \frac{16}{\sqrt{30}}\vec{k}$$

Question 11:

Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

Answer

Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$.

It is observed that $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k} = -2(2\hat{i} - 3\hat{j} + 4\hat{k}) = -2\vec{a}$

$$\therefore \vec{b} = \lambda\vec{a}$$

where,

$$\lambda = -2$$

Hence, the given vectors are collinear.

Question 12:

Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$

Answer

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$.

$$\therefore |\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

Hence, the direction cosines of \vec{a} are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$.

Question 13:

Find the direction cosines of the vector joining the points A (1, 2, -3) and B (-1, -2, 1) directed from A to B.

Answer

The given points are A (1, 2, -3) and B (-1, -2, 1).

$$\therefore \overline{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + \{1-(-3)\}\hat{k}$$

$$\Rightarrow \overline{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$$

$$\therefore |\overline{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4+16+16} = \sqrt{36} = 6$$

Hence, the direction cosines of \overline{AB} are $\left(-\frac{2}{6}, -\frac{4}{6}, \frac{4}{6}\right) = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$.

Question 14:

Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY, and OZ.

Answer

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$.

Then,

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Therefore, the direction cosines of \vec{a} are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Now, let α , β , and γ be the angles formed by \vec{a} with the positive directions of x , y , and z axes.

Then, we have $\cos \alpha = \frac{1}{\sqrt{3}}$, $\cos \beta = \frac{1}{\sqrt{3}}$, $\cos \gamma = \frac{1}{\sqrt{3}}$.

Hence, the given vector is equally inclined to axes OX, OY, and OZ.

Question 15:

Find the position vector of a point R which divides the line joining two points P and Q

whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2:1

(i) internally

(ii) externally

Answer

The position vector of point R dividing the line segment joining two points

P and Q in the ratio $m:n$ is given by:

i. Internally:

$$\frac{m\vec{b} + n\vec{a}}{m + n}$$

ii. Externally:

$$\frac{m\vec{b} - n\vec{a}}{m - n}$$

Position vectors of P and Q are given as:

$$\overline{OP} = \hat{i} + 2\hat{j} - \hat{k} \text{ and } \overline{OQ} = -\hat{i} + \hat{j} + \hat{k}$$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2:1 is given by,

$$\begin{aligned} \overline{OR} &= \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2+1} = \frac{(-2\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})}{3} \\ &= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k} \end{aligned}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2:1 is given by,

$$\begin{aligned} \overline{OR} &= \frac{2(-\hat{i} + \hat{j} + \hat{k}) - 1(\hat{i} + 2\hat{j} - \hat{k})}{2-1} = \frac{(-2\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})}{1} \\ &= -3\hat{i} + 3\hat{k} \end{aligned}$$

Question 16:

Find the position vector of the mid point of the vector joining the points P (2, 3, 4) and Q (4, 1, - 2).

Answer

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, - 2) is given by,

$$\begin{aligned} \overline{OR} &= \frac{(2\hat{i} + 3\hat{j} + 4\hat{k}) + (4\hat{i} + \hat{j} - 2\hat{k})}{2} = \frac{(2+4)\hat{i} + (3+1)\hat{j} + (4-2)\hat{k}}{2} \\ &= \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2} = 3\hat{i} + 2\hat{j} + \hat{k} \end{aligned}$$

Question 17:

Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}$,

$\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right angled triangle.

Answer

Position vectors of points A, B, and C are respectively given as:

$$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}, \vec{b} = 2\hat{i} - \hat{j} + \hat{k} \text{ and } \vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$$

$$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k}, \vec{b} = 2\hat{i} - \hat{j} + \hat{k} \text{ and } \vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$$

$$\therefore \overline{AB} = \vec{b} - \vec{a} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\overline{BC} = \vec{c} - \vec{b} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overline{CA} = \vec{a} - \vec{c} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\therefore |\overline{AB}|^2 = (-1)^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35$$

$$|\overline{BC}|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41$$

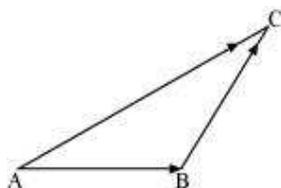
$$|\overline{CA}|^2 = 2^2 + (-1)^2 + 1^2 = 4 + 1 + 1 = 6$$

$$\therefore |\overline{AB}|^2 + |\overline{CA}|^2 = 35 + 6 = 41 = |\overline{BC}|^2$$

Hence, ABC is a right-angled triangle.

Question 18:

In triangle ABC which of the following is **not** true:



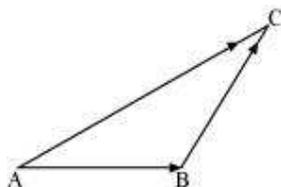
A. $\overline{AB} + \overline{BC} + \overline{CA} = \vec{0}$

B. $\overline{AB} + \overline{BC} - \overline{AC} = \vec{0}$

C. $\overline{AB} + \overline{BC} - \overline{CA} = \vec{0}$

D. $\overline{AB} - \overline{CB} + \overline{CA} = \vec{0}$

Answer



On applying the triangle law of addition in the given triangle, we have:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad \dots(1)$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0} \quad \dots(2)$$

\therefore The equation given in alternative A is true.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$$

\therefore The equation given in alternative B is true.

From equation (2), we have:

$$\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$$

\therefore The equation given in alternative D is true.

Now, consider the equation given in alternative C:

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CA} \quad \dots(3)$$

From equations (1) and (3), we have:

$$\overrightarrow{AC} = \overrightarrow{CA}$$

$$\Rightarrow \overrightarrow{AC} = -\overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AC} + \overrightarrow{AC} = \vec{0}$$

$$\Rightarrow 2\overrightarrow{AC} = \vec{0}$$

$$\Rightarrow \overrightarrow{AC} = \vec{0}, \text{ which is not true.}$$

Hence, the equation given in alternative C is **incorrect**.

The correct answer is **C**.

Question 19:

If \vec{a} and \vec{b} are two collinear vectors, then which of the following are **incorrect**:

- A. $\vec{b} = \lambda\vec{a}$, for some scalar λ
- B. $\vec{a} = \pm\vec{b}$
- C. the respective components of \vec{a} and \vec{b} are proportional
- D. both the vectors \vec{a} and \vec{b} have same direction, but different magnitudes

Answer

If \vec{a} and \vec{b} are two collinear vectors, then they are parallel.

Therefore, we have:

$$\vec{b} = \lambda\vec{a} \text{ (For some scalar } \lambda)$$

If $\lambda = \pm 1$, then $\vec{a} = \pm\vec{b}$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{b} = \lambda\vec{a}.$$

$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

$$\Rightarrow b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$$

$$\Rightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Thus, the respective components of \vec{a} and \vec{b} are proportional.

However, vectors \vec{a} and \vec{b} can have different directions.

Hence, the statement given in **D** is **incorrect**.

The correct answer is **D**.

Exercise 10.3

Question 1:

Find the angle between two vectors \vec{a} and \vec{b} with magnitudes $\sqrt{3}$ and 2, respectively

having $\vec{a} \cdot \vec{b} = \sqrt{6}$.

Answer

It is given that,

$$|\vec{a}| = \sqrt{3}, |\vec{b}| = 2 \text{ and, } \vec{a} \cdot \vec{b} = \sqrt{6}$$

Now, we know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \sqrt{6} = \sqrt{3} \times 2 \times \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\sqrt{6}}{\sqrt{3} \times 2}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, the angle between the given vectors \vec{a} and \vec{b} is $\frac{\pi}{4}$.

Question 2:

Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$

Answer

The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\begin{aligned} \text{Now, } \vec{a} \cdot \vec{b} &= (\hat{i} - 2\hat{j} + 3\hat{k})(3\hat{i} - 2\hat{j} + \hat{k}) \\ &= 1.3 + (-2)(-2) + 3.1 \\ &= 3 + 4 + 3 \\ &= 10 \end{aligned}$$

Also, we know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore 10 = \sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Question 3:

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

Answer

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$.

Now, projection of vector \vec{a} on \vec{b} is given by,

$$\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) = \frac{1}{\sqrt{1+1}}\{1.1 + (-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$$

Hence, the projection of vector \vec{a} on \vec{b} is 0.

Question 4:

Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Answer

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

Now, projection of vector \vec{a} on \vec{b} is given by,

$$\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) = \frac{1}{\sqrt{7^2 + (-1)^2 + 8^2}} \{1(7) + 3(-1) + 7(8)\} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}), \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

Answer

$$\text{Let } \vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k},$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k},$$

$$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}) = \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}.$$

$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a} \cdot \vec{b} = \frac{2}{7} \times \frac{3}{7} + \frac{3}{7} \times \left(-\frac{6}{7}\right) + \frac{6}{7} \times \frac{2}{7} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$$

$$\vec{b} \cdot \vec{c} = \frac{3}{7} \times \frac{6}{7} + \left(-\frac{6}{7}\right) \times \frac{2}{7} + \frac{2}{7} \times \left(-\frac{3}{7}\right) = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$$

$$\vec{c} \cdot \vec{a} = \frac{6}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{3}{7} + \left(-\frac{3}{7}\right) \times \frac{6}{7} = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$$

Hence, the given three vectors are mutually perpendicular to each other.

Question 6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Answer

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$$

$$\Rightarrow \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 8$$

$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8 \quad [|\vec{a}| = 8|\vec{b}|]$$

$$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow 63|\vec{b}|^2 = 8$$

$$\Rightarrow |\vec{b}|^2 = \frac{8}{63}$$

$$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}} \quad [\text{Magnitude of a vector is non-negative}]$$

$$\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}| = 8|\vec{b}| = \frac{8 \times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$$

Question 7:

Evaluate the product $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$.

Answer

$$(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$$

$$= 3\vec{a} \cdot 2\vec{a} + 3\vec{a} \cdot 7\vec{b} - 5\vec{b} \cdot 2\vec{a} - 5\vec{b} \cdot 7\vec{b}$$

$$= 6\vec{a} \cdot \vec{a} + 21\vec{a} \cdot \vec{b} - 10\vec{a} \cdot \vec{b} - 35\vec{b} \cdot \vec{b}$$

$$= 6|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 35|\vec{b}|^2$$

Question 8:

Find the magnitude of two vectors \vec{a} and \vec{b} , having the same magnitude and such that

the angle between them is 60° and their scalar product is $\frac{1}{2}$.

Answer

Let θ be the angle between the vectors \vec{a} and \vec{b} .

It is given that $|\vec{a}| = |\vec{b}|$, $\vec{a} \cdot \vec{b} = \frac{1}{2}$, and $\theta = 60^\circ$ (1)

We know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \frac{1}{2} = |\vec{a}||\vec{a}|\cos 60^\circ \quad [\text{Using (1)}]$$

$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2 = 1$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

Question 9:

Find $|\vec{x}|$, if for a unit vector \vec{a} , $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$.

Answer

$$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$$

$$\Rightarrow \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12 \quad [|\vec{a}| = 1 \text{ as } \vec{a} \text{ is a unit vector}]$$

$$\Rightarrow |\vec{x}|^2 = 13$$

$$\therefore |\vec{x}| = \sqrt{13}$$

Question 10:

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to \vec{c} , then find the value of λ .

Answer

The given vectors are $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$, and $\vec{c} = 3\hat{i} + \hat{j}$.

Now,

$$\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k}) = (2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}$$

If $(\vec{a} + \lambda\vec{b})$ is perpendicular to \vec{c} , then

$$(\vec{a} + \lambda\vec{b}) \cdot \vec{c} = 0.$$

$$\Rightarrow [(2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}] \cdot (3\hat{i} + \hat{j}) = 0$$

$$\Rightarrow (2 - \lambda)3 + (2 + 2\lambda)1 + (3 + \lambda)0 = 0$$

$$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$$

$$\Rightarrow -\lambda + 8 = 0$$

$$\Rightarrow \lambda = 8$$

Hence, the required value of λ is 8.

Question 11:

Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two nonzero vectors \vec{a} and \vec{b}

Answer

$$\begin{aligned} & (|\vec{a}|\vec{b} + |\vec{b}|\vec{a}) \cdot (|\vec{a}|\vec{b} - |\vec{b}|\vec{a}) \\ &= |\vec{a}|^2 \vec{b} \cdot \vec{b} - |\vec{a}||\vec{b}|\vec{b} \cdot \vec{a} + |\vec{b}||\vec{a}|\vec{a} \cdot \vec{b} - |\vec{b}|^2 \vec{a} \cdot \vec{a} \\ &= |\vec{a}|^2 |\vec{b}|^2 - |\vec{b}|^2 |\vec{a}|^2 \\ &= 0 \end{aligned}$$

Hence, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ are perpendicular to each other.

Question 12:

If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, then what can be concluded about the vector \vec{b} ?

Answer

It is given that $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$

Now,

$$\vec{a} \cdot \vec{a} = 0 \Rightarrow |\vec{a}|^2 = 0 \Rightarrow |\vec{a}| = 0$$

$\therefore \vec{a}$ is a zero vector.

Hence, vector \vec{b} satisfying $\vec{a} \cdot \vec{b} = 0$ can be any vector.

Question 14:

If either vector $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \cdot \vec{b} = 0$. But the converse need not be true. Justify your answer with an example.

Answer

Consider $\vec{a} = 2\hat{i} + 4\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 3\hat{j} - 6\hat{k}$.

Then,

$$\vec{a} \cdot \vec{b} = 2 \cdot 3 + 4 \cdot 3 + 3(-6) = 6 + 12 - 18 = 0$$

We now observe that:

$$|\vec{a}| = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$$

$$\therefore \vec{a} \neq \vec{0}$$

$$|\vec{b}| = \sqrt{3^2 + 3^2 + (-6)^2} = \sqrt{54}$$

$$\therefore \vec{b} \neq \vec{0}$$

Hence, the converse of the given statement need not be true.

Question 15:

If the vertices A, B, C of a triangle ABC are (1, 2, 3), (-1, 0, 0), (0, 1, 2), respectively,

then find $\sphericalangle ABC$. [$\sphericalangle ABC$ is the angle between the vectors \overrightarrow{BA} and \overrightarrow{BC}]

Answer

The vertices of $\triangle ABC$ are given as A (1, 2, 3), B (-1, 0, 0), and C (0, 1, 2).

Also, it is given that $\sphericalangle ABC$ is the angle between the vectors \overrightarrow{BA} and \overrightarrow{BC} .

$$\overline{BA} = \{1 - (-1)\}\hat{i} + (2 - 0)\hat{j} + (3 - 0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overline{BC} = \{0 - (-1)\}\hat{i} + (1 - 0)\hat{j} + (2 - 0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \overline{BA} \cdot \overline{BC} = (2\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 2 \times 1 + 2 \times 1 + 3 \times 2 = 2 + 2 + 6 = 10$$

$$|\overline{BA}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4 + 4 + 9} = \sqrt{17}$$

$$|\overline{BC}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

Now, it is known that:

$$\overline{BA} \cdot \overline{BC} = |\overline{BA}| |\overline{BC}| \cos(\angle ABC)$$

$$\therefore 10 = \sqrt{17} \times \sqrt{6} \cos(\angle ABC)$$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17} \times \sqrt{6}}$$

$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Question 16:

Show that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, -1) are collinear.

Answer

The given points are A (1, 2, 7), B (2, 6, 3), and C (3, 10, -1).

$$\therefore \overline{AB} = (2 - 1)\hat{i} + (6 - 2)\hat{j} + (3 - 7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overline{BC} = (3 - 2)\hat{i} + (10 - 6)\hat{j} + (-1 - 3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overline{AC} = (3 - 1)\hat{i} + (10 - 2)\hat{j} + (-1 - 7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$

$$|\overline{AB}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1 + 16 + 16} = \sqrt{33}$$

$$|\overline{BC}| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1 + 16 + 16} = \sqrt{33}$$

$$|\overline{AC}| = \sqrt{2^2 + 8^2 + (-8)^2} = \sqrt{4 + 64 + 64} = \sqrt{132} = 2\sqrt{33}$$

$$\therefore |\overline{AC}| = |\overline{AB}| + |\overline{BC}|$$

Hence, the given points A, B, and C are collinear.

Question 17:

Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle.

Answer

Let vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ be position vectors of points A, B, and C respectively.

$$\text{i.e., } \overline{OA} = 2\hat{i} - \hat{j} + \hat{k}, \overline{OB} = \hat{i} - 3\hat{j} - 5\hat{k} \text{ and } \overline{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$$

Now, vectors \overline{AB} , \overline{BC} , and \overline{AC} represent the sides of ΔABC .

$$\text{i.e., } \overline{OA} = 2\hat{i} - \hat{j} + \hat{k}, \overline{OB} = \hat{i} - 3\hat{j} - 5\hat{k}, \text{ and } \overline{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$$

$$\therefore \overline{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\overline{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\overline{AC} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$|\overline{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$$

$$|\overline{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$|\overline{AC}| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$$

$$\therefore |\overline{BC}|^2 + |\overline{AC}|^2 = 6 + 35 = 41 = |\overline{AB}|^2$$

Hence, ΔABC is a right-angled triangle.

Question 18:

If \vec{a} is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then $\lambda\vec{a}$ is unit vector if

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = \frac{1}{|\lambda|}$

Answer

Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}| = 1$.

Now,

$$|\lambda\vec{a}| = 1$$

$$\Rightarrow |\lambda||\vec{a}| = 1$$

$$\Rightarrow |\vec{a}| = \frac{1}{|\lambda|} \quad [\lambda \neq 0]$$

$$\Rightarrow a = \frac{1}{|\lambda|} \quad [|\vec{a}| = a]$$

Hence, vector $\lambda\vec{a}$ is a unit vector if $a = \frac{1}{|\lambda|}$.

The correct answer is D.

Exercise 10.4

Question 1:

Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Answer

We have,

$$\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k} \text{ and } \vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$$

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} \\ &= \hat{i}(-14+14) - \hat{j}(2-21) + \hat{k}(-2+21) = 19\hat{j} + 19\hat{k} \end{aligned}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{(19)^2 + (19)^2} = \sqrt{2 \times (19)^2} = 19\sqrt{2}$$

Question 2:

Find a unit vector perpendicular to each of the vector $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where

$$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k} \text{ and } \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$$

Answer

We have,

$$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k} \text{ and } \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\therefore \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j}, \quad \vec{a} - \vec{b} = 2\hat{i} + 4\hat{k}$$

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \hat{i}(16) - \hat{j}(16) + \hat{k}(-8) = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\begin{aligned} \therefore |(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| &= \sqrt{16^2 + (-16)^2 + (-8)^2} \\ &= \sqrt{2^2 \times 8^2 + 2^2 \times 8^2 + 8^2} \\ &= 8\sqrt{2^2 + 2^2 + 1} = 8\sqrt{9} = 8 \times 3 = 24 \end{aligned}$$

Hence, the unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by the relation,

$$\begin{aligned} &= \pm \frac{(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})}{|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|} = \pm \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24} \\ &= \pm \frac{2\hat{i} - 2\hat{j} - \hat{k}}{3} = \pm \frac{2}{3}\hat{i} \mp \frac{2}{3}\hat{j} \mp \frac{1}{3}\hat{k} \end{aligned}$$

Question 3:

If a unit vector \vec{a} makes an angle $\frac{\pi}{3}$ with \hat{i} , $\frac{\pi}{4}$ with \hat{j} and an acute angle θ with \hat{k} , then find θ and hence, the components of \vec{a} .

Answer

Let unit vector \vec{a} have (a_1, a_2, a_3) components.

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

Since \vec{a} is a unit vector, $|\vec{a}| = 1$.

Also, it is given that \vec{a} makes angles $\frac{\pi}{3}$ with \hat{i} , $\frac{\pi}{4}$ with \hat{j} , and an acute angle θ with \hat{k} .

Then, we have:

$$\cos \frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{2} = a_1 \quad [|\vec{a}| = 1]$$

$$\cos \frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \quad [|\vec{a}| = 1]$$

$$\text{Also, } \cos \theta = \frac{a_3}{|\vec{a}|}$$

$$\Rightarrow a_3 = \cos \theta$$

Now,

$$|a| = 1$$

$$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2 \theta = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2 \theta = 1$$

$$\Rightarrow \frac{3}{4} + \cos^2 \theta = 1$$

$$\Rightarrow \cos^2 \theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_3 = \cos \frac{\pi}{3} = \frac{1}{2}$$

Hence, $\theta = \frac{\pi}{3}$ and the components of \vec{a} are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Question 4:

Show that

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$$

Answer

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$$

$$= (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b}$$

[By distributivity of vector product over addition]

$$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$$

[Again, by distributivity of vector product over addition]

$$= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$$

$$= 2\vec{a} \times \vec{b}$$

Question 5:

Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

Answer

$$(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\Rightarrow \hat{i}(6\mu - 27\lambda) - \hat{j}(2\mu - 27) + \hat{k}(2\lambda - 6) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

On comparing the corresponding components, we have:

$$6\mu - 27\lambda = 0$$

$$2\mu - 27 = 0$$

$$2\lambda - 6 = 0$$

Now,

$$2\lambda - 6 = 0 \Rightarrow \lambda = 3$$

$$2\mu - 27 = 0 \Rightarrow \mu = \frac{27}{2}$$

Hence, $\lambda = 3$ and $\mu = \frac{27}{2}$.

Question 6:

Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you conclude about the vectors \vec{a} and \vec{b} ?

Answer

$$\vec{a} \cdot \vec{b} = 0$$

Then,

(i) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$, or $\vec{a} \perp \vec{b}$ (in case \vec{a} and \vec{b} are non-zero)

$$\vec{a} \times \vec{b} = \vec{0}$$

(ii) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$, or $\vec{a} \parallel \vec{b}$ (in case \vec{a} and \vec{b} are non-zero)

But, \vec{a} and \vec{b} cannot be perpendicular and parallel simultaneously.

Hence, $|\vec{a}| = 0$ or $|\vec{b}| = 0$.

Question 7:

Let the vectors \vec{a} , \vec{b} , \vec{c} given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show

$$\text{that } \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Answer

We have,

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$(\vec{b} + \vec{c}) = (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$$

$$\text{Now, } \vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$\begin{aligned} &= \hat{i}[a_2(b_3 + c_3) - a_3(b_2 + c_2)] - \hat{j}[a_1(b_3 + c_3) - a_3(b_1 + c_1)] + \hat{k}[a_1(b_2 + c_2) - a_2(b_1 + c_1)] \\ &= \hat{i}[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2] + \hat{j}[-a_1b_3 - a_1c_3 + a_3b_1 + a_3c_1] + \hat{k}[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1] \quad \dots(1) \end{aligned}$$

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{i}[a_2b_3 - a_3b_2] + \hat{j}[b_1a_3 - a_1b_3] + \hat{k}[a_1b_2 - a_2b_1] \quad (2) \end{aligned}$$

$$\begin{aligned} \vec{a} \times \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \hat{i}[a_2c_3 - a_3c_2] + \hat{j}[a_3c_1 - a_1c_3] + \hat{k}[a_1c_2 - a_2c_1] \quad (3) \end{aligned}$$

On adding (2) and (3), we get:

$$\begin{aligned} (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) &= \hat{i}[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2] + \hat{j}[b_1a_3 + a_3c_1 - a_1b_3 - a_1c_3] \\ &+ \hat{k}[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1] \quad (4) \end{aligned}$$

Now, from (1) and (4), we have:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Hence, the given result is proved.

Question 8:

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Answer

Take any parallel non-zero vectors so that $\vec{a} \times \vec{b} = \vec{0}$.

$$\text{Let } \vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}.$$

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} = \hat{i}(24 - 24) - \hat{j}(16 - 16) + \hat{k}(12 - 12) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

It can now be observed that:

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$$\therefore \vec{a} \neq \vec{0}$$

$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}$$

$$\therefore \vec{b} \neq \vec{0}$$

Hence, the converse of the given statement need not be true.

Question 9:

Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Answer

The vertices of triangle ABC are given as A (1, 1, 2), B (2, 3, 5), and C (1, 5, 5).

The adjacent sides \overline{AB} and \overline{BC} of ΔABC are given as:

$$\overline{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overline{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$$

$$\text{Area of } \Delta ABC = \frac{1}{2} |\overline{AB} \times \overline{BC}|$$

$$\overline{AB} \times \overline{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore |\overline{AB} \times \overline{BC}| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector

$$\vec{a} = \hat{i} - \hat{j} + 3\hat{k} \text{ and } \vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}.$$

Answer

The area of the parallelogram whose adjacent sides are \vec{a} and \vec{b} is $|\vec{a} \times \vec{b}|$.

Adjacent sides are given as:

$$\vec{a} = \hat{i} - \hat{j} + 3\hat{k} \text{ and } \vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix} = \hat{i}(-1+21) - \hat{j}(1-6) + \hat{k}(-7+2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{20^2 + 5^2 + 5^2} = \sqrt{400 + 25 + 25} = 15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units.

Question 11:

Let the vectors \vec{a} and \vec{b} be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if

the angle between \vec{a} and \vec{b} is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Answer

It is given that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$.

We know that $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\hat{n}$, where \hat{n} is a unit vector perpendicular to both \vec{a} and \vec{b} and θ is the angle between \vec{a} and \vec{b} .

Now, $\vec{a} \times \vec{b}$ is a unit vector if $|\vec{a} \times \vec{b}| = 1$.

$$|\vec{a} \times \vec{b}| = 1$$

$$\Rightarrow |\vec{a}||\vec{b}|\sin\theta = 1$$

$$\Rightarrow |\vec{a}||\vec{b}|\sin\theta = 1$$

$$\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times \sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, $\vec{a} \times \vec{b}$ is a unit vector if the angle between \vec{a} and \vec{b} is $\frac{\pi}{4}$.

The correct answer is B.

Question 12:

Area of a rectangle having vertices A, B, C, and D with position vectors

$$-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k} \text{ and } -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k} \text{ respectively is}$$

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Answer

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overline{OA} = -\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \quad \overline{OB} = \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}, \quad \overline{OC} = \hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}, \quad \overline{OD} = -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$$

The adjacent sides \overline{AB} and \overline{BC} of the given rectangle are given as:

$$\overline{AB} = (1+1)\hat{i} + \left(\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = 2\hat{i}$$

$$\overline{BC} = (1-1)\hat{i} + \left(-\frac{1}{2} - \frac{1}{2}\right)\hat{j} + (4-4)\hat{k} = -\hat{j}$$

$$\therefore \overline{AB} \times \overline{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \hat{k}(-2) = -2\hat{k}$$

$$|\overline{AB} \times \overline{BC}| = \sqrt{(-2)^2} = 2$$

Now, it is known that the area of a parallelogram whose adjacent sides are

$$\vec{a} \text{ and } \vec{b} \text{ is } |\vec{a} \times \vec{b}|.$$

Hence, the area of the given rectangle is $|\overline{AB} \times \overline{BC}| = 2$ square units.

The correct answer is C.

Miscellaneous Solutions

Question 1:

Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

Answer

If \vec{r} is a unit vector in the XY-plane, then $\vec{r} = \cos \theta \hat{i} + \sin \theta \hat{j}$.

Here, θ is the angle made by the unit vector with the positive direction of the x-axis.

Therefore, for $\theta = 30^\circ$:

$$\vec{r} = \cos 30^\circ \hat{i} + \sin 30^\circ \hat{j} = \frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j}$$

Hence, the required unit vector is $\frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j}$

Question 2:

Find the scalar components and magnitude of the vector joining the points

$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Answer

The vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ can be obtained by,

\overline{PQ} = Position vector of Q – Position vector of P

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$|\overline{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Hence, the scalar components and the magnitude of the vector joining the given points

are respectively $\{(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)\}$ and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

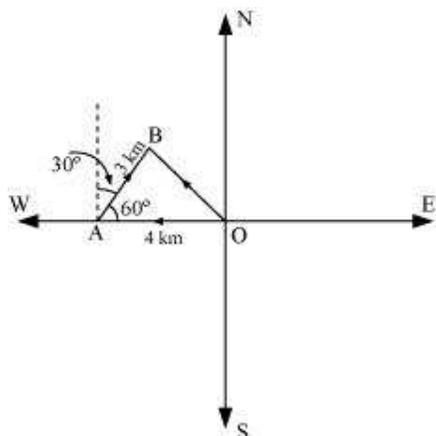
Question 3:

A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Answer

Let O and B be the initial and final positions of the girl respectively.

Then, the girl's position can be shown as:



Now, we have:

$$\overline{OA} = -4\hat{i}$$

$$\overline{AB} = \hat{i}|\overline{AB}|\cos 60^\circ + \hat{j}|\overline{AB}|\sin 60^\circ$$

$$= \hat{i}3 \times \frac{1}{2} + \hat{j}3 \times \frac{\sqrt{3}}{2}$$

$$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

By the triangle law of vector addition, we have:

$$\overline{OB} = \overline{OA} + \overline{AB}$$

$$= (-4\hat{i}) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right)$$

$$= \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$= \left(\frac{-8+3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

Hence, the girl's displacement from her initial point of departure is

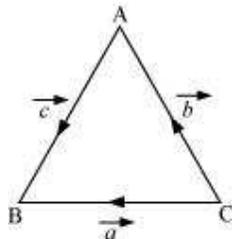
$$\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

Question 4:

If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Answer

In $\triangle ABC$, let $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, and $\overrightarrow{AB} = \vec{c}$ (as shown in the following figure).



Now, by the triangle law of vector addition, we have $\vec{a} = \vec{b} + \vec{c}$.

It is clearly known that $|\vec{a}|$, $|\vec{b}|$, and $|\vec{c}|$ represent the sides of $\triangle ABC$.

Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$$\therefore |\vec{a}| < |\vec{b}| + |\vec{c}|$$

Hence, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$.

Question 5:

Find the value of x for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector.

Answer

$$x(\hat{i} + \hat{j} + \hat{k}) \text{ is a unit vector if } |x(\hat{i} + \hat{j} + \hat{k})| = 1$$

Now,

$$\begin{aligned} |x(\hat{i} + \hat{j} + \hat{k})| &= 1 \\ \Rightarrow \sqrt{x^2 + x^2 + x^2} &= 1 \\ \Rightarrow \sqrt{3x^2} &= 1 \\ \Rightarrow \sqrt{3}x &= 1 \\ \Rightarrow x &= \pm \frac{1}{\sqrt{3}} \end{aligned}$$

Hence, the required value of x is $\pm \frac{1}{\sqrt{3}}$.

Question 6:

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \text{ and } \vec{b} = \hat{i} - 2\hat{j} + \hat{k}.$$

Answer

We have,

$$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \text{ and } \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$$

Let \vec{c} be the resultant of \vec{a} and \vec{b} .

Then,

$$\vec{c} = \vec{a} + \vec{b} = (2+1)\hat{i} + (3-2)\hat{j} + (-1+1)\hat{k} = 3\hat{i} + \hat{j}$$

$$\therefore |\vec{c}| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$$

$$\therefore \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{(3\hat{i} + \hat{j})}{\sqrt{10}}$$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors \vec{a} and \vec{b} is

$$\pm 5 \cdot \hat{c} = \pm 5 \cdot \frac{1}{\sqrt{10}}(3\hat{i} + \hat{j}) = \pm \frac{3\sqrt{10}\hat{i}}{2} \pm \frac{\sqrt{10}\hat{j}}{2}.$$

Question 7:

If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.

Answer

We have,

$$\begin{aligned}\vec{a} &= \hat{i} + \hat{j} + \hat{k}, \vec{b} = 2\hat{i} - \hat{j} + 3\hat{k} \text{ and } \vec{c} = \hat{i} - 2\hat{j} + \hat{k} \\ 2\vec{a} - \vec{b} + 3\vec{c} &= 2(\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k}) \\ &= 2\hat{i} + 2\hat{j} + 2\hat{k} - 2\hat{i} + \hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k} \\ &= 3\hat{i} - 3\hat{j} + 2\hat{k}\end{aligned}$$

$$|2\vec{a} - \vec{b} + 3\vec{c}| = \sqrt{3^2 + (-3)^2 + 2^2} = \sqrt{9+9+4} = \sqrt{22}$$

Hence, the unit vector along $2\vec{a} - \vec{b} + 3\vec{c}$ is

$$\frac{2\vec{a} - \vec{b} + 3\vec{c}}{|2\vec{a} - \vec{b} + 3\vec{c}|} = \frac{3\hat{i} - 3\hat{j} + 2\hat{k}}{\sqrt{22}} = \frac{3}{\sqrt{22}}\hat{i} - \frac{3}{\sqrt{22}}\hat{j} + \frac{2}{\sqrt{22}}\hat{k}.$$

Question 8:

Show that the points A (1, -2, -8), B (5, 0, -2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

Answer

The given points are A (1, -2, -8), B (5, 0, -2), and C (11, 3, 7).

$$\therefore \vec{AB} = (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\vec{BC} = (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\vec{AC} = (11-1)\hat{i} + (3+2)\hat{j} + (7+8)\hat{k} = 10\hat{i} + 5\hat{j} + 15\hat{k}$$

$$|\vec{AB}| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{16+4+36} = \sqrt{56} = 2\sqrt{14}$$

$$|\vec{BC}| = \sqrt{6^2 + 3^2 + 9^2} = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}$$

$$|\vec{AC}| = \sqrt{10^2 + 5^2 + 15^2} = \sqrt{100+25+225} = \sqrt{350} = 5\sqrt{14}$$

$$\therefore |\vec{AC}| = |\vec{AB}| + |\vec{BC}|$$

Thus, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda:1$. Then, we have:

$$\overline{OB} = \frac{\lambda\overline{OC} + \overline{OA}}{(\lambda+1)}$$

$$\Rightarrow 5\hat{i} - 2\hat{k} = \frac{\lambda(11\hat{i} + 3\hat{j} + 7\hat{k}) + (\hat{i} - 2\hat{j} - 8\hat{k})}{\lambda+1}$$

$$\Rightarrow (\lambda+1)(5\hat{i} - 2\hat{k}) = 11\lambda\hat{i} + 3\lambda\hat{j} + 7\lambda\hat{k} + \hat{i} - 2\hat{j} - 8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i} - 2(\lambda+1)\hat{k} = (11\lambda+1)\hat{i} + (3\lambda-2)\hat{j} + (7\lambda-8)\hat{k}$$

On equating the corresponding components, we get:

$$5(\lambda+1) = 11\lambda+1$$

$$\Rightarrow 5\lambda+5 = 11\lambda+1$$

$$\Rightarrow 6\lambda = 4$$

$$\Rightarrow \lambda = \frac{4}{6} = \frac{2}{3}$$

Hence, point B divides AC in the ratio 2:3.

Question 9:

Find the position vector of a point R which divides the line joining two points P and Q

whose position vectors are $(2\vec{a} + \vec{b})$ and $(\vec{a} - 3\vec{b})$ externally in the ratio 1: 2. Also, show that P is the mid point of the line segment RQ.

Answer

It is given that $\overline{OP} = 2\vec{a} + \vec{b}$, $\overline{OQ} = \vec{a} - 3\vec{b}$.

It is given that point R divides a line segment joining two points P and Q externally in the ratio 1: 2. Then, on using the section formula, we get:

$$\overline{OR} = \frac{2(2\vec{a} + \vec{b}) - (\vec{a} - 3\vec{b})}{2-1} = \frac{4\vec{a} + 2\vec{b} - \vec{a} + 3\vec{b}}{1} = 3\vec{a} + 5\vec{b}$$

Therefore, the position vector of point R is $3\vec{a} + 5\vec{b}$.

$$\text{Position vector of the mid-point of RQ} = \frac{\overline{OQ} + \overline{OR}}{2}$$

$$\begin{aligned}
 &= \frac{(\vec{a} - 3\vec{b}) + (3\vec{a} + 5\vec{b})}{2} \\
 &= 2\vec{a} + \vec{b} \\
 &= \overline{OP}
 \end{aligned}$$

Hence, P is the mid-point of the line segment RQ.

Question 10:

The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$.

Find the unit vector parallel to its diagonal. Also, find its area.

Answer

Adjacent sides of a parallelogram are given as: $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$

Then, the diagonal of a parallelogram is given by $\vec{a} + \vec{b}$.

$$\vec{a} + \vec{b} = (2+1)\hat{i} + (-4-2)\hat{j} + (5-3)\hat{k} = 3\hat{i} - 6\hat{j} + 2\hat{k}$$

Thus, the unit vector parallel to the diagonal is

$$\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + (-6)^2 + 2^2}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{\sqrt{9 + 36 + 4}} = \frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7} = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k}.$$

\therefore Area of parallelogram ABCD = $|\vec{a} \times \vec{b}|$

$$\begin{aligned}
 \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} \\
 &= \hat{i}(12+10) - \hat{j}(-6-5) + \hat{k}(-4+4) \\
 &= 22\hat{i} + 11\hat{j} \\
 &= 11(2\hat{i} + \hat{j})
 \end{aligned}$$

$$\therefore |\vec{a} \times \vec{b}| = 11\sqrt{2^2 + 1^2} = 11\sqrt{5}$$

Hence, the area of the parallelogram is $11\sqrt{5}$ square units.

Question 11:

Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ

are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Answer

Let a vector be equally inclined to axes OX, OY, and OZ at angle α .

Then, the direction cosines of the vector are $\cos \alpha$, $\cos \alpha$, and $\cos \alpha$.

Now,

$$\cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$$

$$\Rightarrow 3 \cos^2 \alpha = 1$$

$$\Rightarrow \cos \alpha = \frac{1}{\sqrt{3}}$$

Hence, the direction cosines of the vector which are equally inclined to the axes

are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Question 12:

Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector \vec{d} which is perpendicular to both \vec{a} and \vec{b} , and $\vec{c} \cdot \vec{d} = 15$.

Answer

$$\text{Let } \vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}.$$

Since \vec{d} is perpendicular to both \vec{a} and \vec{b} , we have:

$$\vec{d} \cdot \vec{a} = 0$$

$$\Rightarrow d_1 + 4d_2 + 2d_3 = 0 \quad \dots(\text{i})$$

And,

$$\vec{d} \cdot \vec{b} = 0$$

$$\Rightarrow 3d_1 - 2d_2 + 7d_3 = 0 \quad \dots(\text{ii})$$

Also, it is given that:

$$\vec{c} \cdot \vec{d} = 15$$

$$\Rightarrow 2d_1 - d_2 + 4d_3 = 15 \quad \dots(\text{iii})$$

On solving (i), (ii), and (iii), we get:

$$d_1 = \frac{160}{3}, d_2 = -\frac{5}{3} \text{ and } d_3 = -\frac{70}{3}$$

$$\therefore \vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k} = \frac{1}{3}(160\hat{i} - 5\hat{j} - 70\hat{k})$$

Hence, the required vector is $\frac{1}{3}(160\hat{i} - 5\hat{j} - 70\hat{k})$.

Question 13:

The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors

$2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ .

Answer

$$(2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

Therefore, unit vector along $(2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k})$ is given as:

$$\frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2 + \lambda)^2 + 6^2 + (-2)^2}} = \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{4 + 4\lambda + \lambda^2 + 36 + 4}} = \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

Scalar product of $(\hat{i} + \hat{j} + \hat{k})$ with this unit vector is 1.

$$\Rightarrow (\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow \frac{(2 + \lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow \sqrt{\lambda^2 + 4\lambda + 44} = \lambda + 6$$

$$\Rightarrow \lambda^2 + 4\lambda + 44 = (\lambda + 6)^2$$

$$\Rightarrow \lambda^2 + 4\lambda + 44 = \lambda^2 + 12\lambda + 36$$

$$\Rightarrow 8\lambda = 8$$

$$\Rightarrow \lambda = 1$$

Hence, the value of λ is 1.

Question 14:

If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to \vec{a}, \vec{b} and \vec{c} .

Answer

Since \vec{a}, \vec{b} , and \vec{c} are mutually perpendicular vectors, we have

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0.$$

It is given that:

$$|\vec{a}| = |\vec{b}| = |\vec{c}|$$

Let vector $\vec{a} + \vec{b} + \vec{c}$ be inclined to \vec{a}, \vec{b} , and \vec{c} at angles θ_1, θ_2 , and θ_3 respectively.

Then, we have:

$$\begin{aligned} \cos \theta_1 &= \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} = \frac{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} \\ &= \frac{|\vec{a}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{a}|} \quad [\vec{b} \cdot \vec{a} = \vec{c} \cdot \vec{a} = 0] \end{aligned}$$

$$\begin{aligned} \cos \theta_2 &= \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{b}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} = \frac{\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{b}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} \\ &= \frac{|\vec{b}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{b}|} \quad [\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{b} = 0] \end{aligned}$$

$$\begin{aligned} \cos \theta_3 &= \frac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{c}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} = \frac{\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c}}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} \\ &= \frac{|\vec{c}|^2}{|\vec{a} + \vec{b} + \vec{c}| |\vec{c}|} \quad [\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c} = 0] \\ &= \frac{|\vec{c}|}{|\vec{a} + \vec{b} + \vec{c}|} \end{aligned}$$

Now, as $|\vec{a}| = |\vec{b}| = |\vec{c}|$, $\cos \theta_1 = \cos \theta_2 = \cos \theta_3$.
 $\therefore \theta_1 = \theta_2 = \theta_3$

Hence, the vector $(\vec{a} + \vec{b} + \vec{c})$ is equally inclined to \vec{a}, \vec{b} , and \vec{c} .

Question 15:

Prove that $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$, if and only if \vec{a}, \vec{b} are perpendicular,
 given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$.

Answer

$$\begin{aligned} (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) &= |\vec{a}|^2 + |\vec{b}|^2 \\ \Leftrightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} &= |\vec{a}|^2 + |\vec{b}|^2 \quad [\text{Distributivity of scalar products over addition}] \\ \Leftrightarrow |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 \quad [\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \text{ (Scalar product is commutative)}] \\ \Leftrightarrow 2\vec{a} \cdot \vec{b} &= 0 \\ \Leftrightarrow \vec{a} \cdot \vec{b} &= 0 \\ \therefore \vec{a} \text{ and } \vec{b} \text{ are perpendicular.} & \quad [\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0} \text{ (Given)}] \end{aligned}$$

Question 16:

If θ is the angle between two vectors \vec{a} and \vec{b} , then $\vec{a} \cdot \vec{b} \geq 0$ only when

(A) $0 < \theta < \frac{\pi}{2}$ (B) $0 \leq \theta \leq \frac{\pi}{2}$

(C) $0 < \theta < \pi$ (D) $0 \leq \theta \leq \pi$

Answer

Let θ be the angle between two vectors \vec{a} and \vec{b} .

Then, without loss of generality, \vec{a} and \vec{b} are non-zero vectors so

that $|\vec{a}|$ and $|\vec{b}|$ are positive.

It is known that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \vec{a} \cdot \vec{b} \geq 0$$

$$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$$

$$\Rightarrow \cos\theta \geq 0 \quad [|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}]$$

$$\Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$$

Hence, $\vec{a} \cdot \vec{b} \geq 0$ when $0 \leq \theta \leq \frac{\pi}{2}$.

The correct answer is B.

Question 17:

Let \vec{a} and \vec{b} be two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$

Answer

Let \vec{a} and \vec{b} be two unit vectors and θ be the angle between them.

Then, $|\vec{a}| = |\vec{b}| = 1$.

Now, $\vec{a} + \vec{b}$ is a unit vector if $|\vec{a} + \vec{b}| = 1$.

$$\begin{aligned}
|\vec{a} + \vec{b}| &= 1 \\
\Rightarrow (\vec{a} + \vec{b})^2 &= 1 \\
\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) &= 1 \\
\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} &= 1 \\
\Rightarrow |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 &= 1 \\
\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 &= 1 \\
\Rightarrow 1 + 2 \cdot 1 \cdot 1 \cos\theta + 1 &= 1 \\
\Rightarrow \cos\theta &= -\frac{1}{2} \\
\Rightarrow \theta &= \frac{2\pi}{3}
\end{aligned}$$

Hence, $\vec{a} + \vec{b}$ is a unit vector if $\theta = \frac{2\pi}{3}$.

The correct answer is D.

Question 18:

The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$ is

(A) 0 (B) -1 (C) 1 (D) 3

Answer

$$\begin{aligned}
&\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j}) \\
&= \hat{i} \cdot \hat{i} + \hat{j} \cdot (-\hat{j}) + \hat{k} \cdot \hat{k} \\
&= 1 - \hat{j} \cdot \hat{j} + 1 \\
&= 1 - 1 + 1 \\
&= 1
\end{aligned}$$

The correct answer is C.

Question 19:

If θ is the angle between any two vectors \vec{a} and \vec{b} , then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Answer

Let θ be the angle between two vectors \vec{a} and \vec{b} .

Then, without loss of generality, \vec{a} and \vec{b} are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive

$$|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$$

$$\Rightarrow |\vec{a}| |\vec{b}| \cos \theta = |\vec{a}| |\vec{b}| \sin \theta$$

$$\Rightarrow \cos \theta = \sin \theta \quad [|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}]$$

$$\Rightarrow \tan \theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to $\frac{\pi}{4}$.

The correct answer is B.